Differentiable Manifolds

1. Let $\Phi: M \to N$ be a submanifold where $\dim(M) > 1$ and let

$$\Phi^*: C^\infty(N, \mathbb{R}) \to C^\infty(M, \mathbb{R})$$

be the restriction map $f \mapsto f \circ \Phi$.

a) Show that in general $\Phi^*$ is neither injective nor surjective.

b) Prove that if $\Phi$ is a closed imbedding then $\Phi^*$ is surjective.

2. Consider the vector field $v = \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ on $\mathbb{R}^2$.

a) Find the integral curve of $v$ through $(a, b) \in \mathbb{R}^2$.

b) Find a smooth map $\mathbb{R}^2 \to \mathbb{R}$ such that the fibers are given by the integral curves of $v$.

c) Find a 1-form $w$ which annihilates $v$. Is $w$ exact?

3. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere with its standard smooth manifold structure. For vectors $a, b \in \mathbb{R}^3$, let $a \times b$ and $\langle a, b \rangle$ respectively denote the vector product and the inner product.

a) Let $n$ be the outward normal vector on $S^2$. Given $\sigma \in \Lambda^1(S^2)$ defined by

$$\sigma(X) = \langle [1, 1, 1], X \times n \rangle$$

prove that $\sigma = i^*(\Sigma)$ where $i: S^2 \to \mathbb{R}^3$ is the identity imbedding and

$$\Sigma = (y - z)dx + (z - x)dy + (x - y)dz.$$

b) Find $\Omega \in \Lambda^2(\mathbb{R}^3)$ such that the volume element $w \in \Lambda^2(S^2)$ can be written in the form $w = i^*(\Omega)$.

c) Does there exist $\theta \in \Lambda^1(\mathbb{R}^3)$ such that $w = i^*(d\theta)$? Explain.
4. True or false? Explain (give a counter example if appropriate).
   a) There exists no compact smooth 2-manifold $M$ which admits an immersion $M \to \mathbb{R}^2$.
   b) Let $M$ be the compact surface and $\Gamma$ be the oriented curve given in the figure. If $\omega$ is a 1-form such that $\int_\Gamma \omega \neq 0$, then $\omega$ is not a closed form.
   
   c) Let $M, N$ be smooth manifolds with $\dim(N) > \dim(M)$ and let $\Phi : N \to M$ be a non-constant smooth map. If for some $y \in M$ the set $\Phi^{-1}(y)$ is a smooth submanifold of $N$, then $y$ is a regular value of $\Phi$. 