M.E.T.U

Department of Mathematics

Preliminary Exam - Sep. 2011

Geometry

Duration : 3 hr.

Each question is 25 pt.

1. **a)** Let $\omega = (x+y)\ dx \wedge dy$, a 2-form on $\mathbb{R}^2$, and $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(r,s,t) = (r - t + s, e^r + t)$. Calculate $f^*(\omega)$, the pullback of ω by f.

   b) Consider the vector field on the plane

   $$X = 2\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}.$$

   Calculate $X(g)$ for any smooth function $g : \mathbb{R}^2 \to \mathbb{R}$.

   c) Calculate the bracket of the vector fields, $[X, Y]$, where $X = 2\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}$ and $Y = e^y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

2. **a)** Consider the real projective plane as the quotient space

   $$P : S^2 \to \mathbb{R}P^2 = S^2/\sim,\ (x,y,z) \mapsto [x : y : z],$$

   where $\sim$ is the equivalence relation on the unit two sphere S^2 defined by, $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ if and only if $(x_1, y_1, z_1) = -(x_2, y_2, z_2)$. Show that

   $$F : \mathbb{R}P^2 \to \mathbb{R}^5,\ [x : y : z] \mapsto (x^2, y^2, xy, yz, zx),$$

   is a smooth embedding.

   b) Let $\sigma : S^2 \to S^2$ be the antipodal map given by

   $$\sigma(x,y,x) = -(x,y,z).$$

Show that for the above map $P : S^2 \to \mathbb{R}P^2$ we have $P = P \circ \sigma$. Let $\omega = x\ dy \wedge dz + y\ dz \wedge dx + z\ dx \wedge dy$ a 2-form on S^2. Prove that $\omega \neq P^*(\nu)$, for any 2-form ν on the real projective plane.

3. **a)** Let $f : K \to \mathbb{R}^n$ and $g : L \to \mathbb{R}^n$ be embeddings of smooth manifolds, so that $\dim K + \dim L < n$. Consider the smooth mapping

$$\phi : K \times L \to \mathbb{R}^n,\ (p,q) \mapsto f(p) - g(q),\ (p,q) \in K \times L\ .$$

Show that a vector $v \in \mathbb{R}^n$ is a regular value for ϕ is and only if the images of the maps $f : K \to \mathbb{R}^n$ and

$$g + v : L \to \mathbb{R}^n,\ q \mapsto g(q) + v$$

are disjoint.

b) Let $f : S^1 \to \mathbb{R}^3$ and $g : S^1 \to \mathbb{R}^3$ be embeddings of the circle into $\mathbb{R}^3$. Using Part (a) conclude that for any $\epsilon > 0$ there is a vector $v \in \mathbb{R}^3$ with $\|v\| < \epsilon$, so that the embedded circles $f(S^1)$ and

$$g(S^1) + v = \{g(q) + v \mid q \in S^1\}$$

are disjoint.

4. A two-form ω on an oriented smooth four manifold, M^4, is called symplectic if it is both closed, $d\omega = 0$, and satisfies

$$(\omega \wedge \omega)(p)(e_1, e_2, e_3, e_4) > 0,$$

at any point $p \in M$, where e_i, $i = 1, 2, 3, 4$, are any set ordered basis (giving the chosen orientation of the manifold) vectors in T_pM^4.

a) Show that the two form $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ is a symplectic form on $\mathbb{R}^4$.

b) Show that the above form satisfies $\omega = d\alpha$, for the 1-form

$$\alpha = x_1\ dx_2 + x_3\ dx_4\ .$$

c) Show that a symplectic form ν on a compact oriented four dimensional manifold, M^4, cannot be an exact form (Hint: Use Stokes theorem).