1. a) Let $\omega = (x + y) \, dx \wedge dy$, a 2-form on $\mathbb{R}^2$, and $f : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $f(r, s, t) = (r - t + s, e^r + t)$. Calculate $f^*(\omega)$, the pullback of $\omega$ by $f$.

b) Consider the vector field on the plane

$$X = 2 \frac{\partial}{\partial x} - xy \frac{\partial}{\partial y}.$$ 

Calculate $X(g)$ for any smooth function $g : \mathbb{R}^2 \to \mathbb{R}$.

c) Calculate the bracket of the vector fields, $[X, Y]$, where

$$X = 2 \frac{\partial}{\partial x} - xy \frac{\partial}{\partial y} \quad \text{and} \quad Y = e^y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}.$$ 

2. a) Consider the real projective plane as the quotient space

$$P : S^2 \to \mathbb{R}P^2 = S^2 / \sim, \quad (x, y, z) \mapsto [x : y : z],$$

where $\sim$ is the equivalence relation on the unit two sphere $S^2$ defined by, $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ if and only if $(x_1, y_1, z_1) = -(x_2, y_2, z_2)$. Show that

$$F : \mathbb{R}P^2 \to \mathbb{R}^5, \quad [x : y : z] \mapsto (x^2, y^2, xy, yz, zx),$$

is a smooth embedding.

b) Let $\sigma : S^2 \to S^2$ be the antipodal map given by

$$\sigma(x, y, z) = -(x, y, z).$$
Show that for the above map \( P : S^2 \to \mathbb{R}P^2 \) we have \( P = P \circ \sigma \). Let 
\[ \omega = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy \]
\( \) a 2-form on \( S^2 \). Prove that 
\( \omega \neq P^*(\nu) \), for any 2-form \( \nu \) on the real projective plane.

3. a) Let \( f : K \to \mathbb{R}^n \) and \( g : L \to \mathbb{R}^n \) be embeddings of smooth manifolds, so that \( \dim K + \dim L < n \). Consider the smooth mapping 
\[ \phi : K \times L \to \mathbb{R}^n, \quad (p, q) \mapsto f(p) - g(q), \quad (p, q) \in K \times L \] .
Show that a vector \( v \in \mathbb{R}^n \) is a regular value for \( \phi \) if and only if the images of the maps \( f : K \to \mathbb{R}^n \) and 
\[ g + v : L \to \mathbb{R}^n, \quad q \mapsto g(q) + v \]
are disjoint.

b) Let \( f : S^1 \to \mathbb{R}^3 \) and \( g : S^1 \to \mathbb{R}^3 \) be embeddings of the circle into \( \mathbb{R}^3 \). Using Part (a) conclude that for any \( \epsilon > 0 \) there is a vector \( v \in \mathbb{R}^3 \) with \( \|v\| < \epsilon \), so that the embedded circles \( f(S^1) \) and 
\[ g(S^1) + v = \{g(q) + v \mid q \in S^1\} \]
are disjoint.

4. A two-form \( \omega \) on an oriented smooth four manifold, \( M^4 \), is called symplectic if it is both closed, \( d\omega = 0 \), and satisfies 
\[ (\omega \wedge \omega)(p)(e_1, e_2, e_3, e_4) > 0, \]
at any point \( p \in M \), where \( e_i, \ i = 1, 2, 3, 4 \), are any set ordered basis (giving the chosen orientation of the manifold) vectors in \( T_p M^4 \).

a) Show that the two form \( \omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \) is a symplectic form on \( \mathbb{R}^4 \).

b) Show that the above form satisfies \( \omega = d\alpha \), for the 1-form 
\[ \alpha = x_1 \, dx_2 + x_3 \, dx_4 \] .

c) Show that a symplectic form \( \nu \) on a compact oriented four dimensional manifold, \( M^4 \), cannot be an exact form (Hint: Use Stokes theorem).