METU MATHEMATICS DEPARTMENT
DIFFERENTIABLE MANIFOLDS
SEPTEMBER 2012 - TMS EXAM

SEPTEMBER 17, 2012

1.) Let $f : \mathbb{R}^3 \to \mathbb{R}$ by

$$f(x, y, z) = (x^2 + y^2 + z^2 - r^2 + 1)^2 - 4(x^2 + y^2) ,$$

where $0 < r < 1$ is a constant.

a) Show that $M = f^{-1}(0)$ is a smooth submanifold of $\mathbb{R}^3$.

b) Determine the tangent space $T_{(r+1,0,0)}M$ as a subspace of $T_{(r+1,0,0)}\mathbb{R}^3$.

2.) Consider the vector field on $\mathbb{R}^3$ given by

$$Y = (z - y)\frac{\partial}{\partial x} + (x - z)\frac{\partial}{\partial y} + (y - x)\frac{\partial}{\partial z} .$$

a) Show that the restriction of Y to the unit sphere $S^2 \subseteq \mathbb{R}^3$ defines a vector field on the unit sphere.

b) Determine the zeros of the vector field on the sphere.

3.) Consider the quotient topological space

$$M = \mathbb{R}^3 \ / \ (x, y, z) \sim (x + 1, y - 1, -z) \ , (x, y, z) \in \mathbb{R}^3 .$$

a) Show that M is a smooth manifold of dimension three.

b) Prove that M is not orientable showing that any 3-form on M has at least one zero.

4.a) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be smooth functions. Show that the 1-form

$$\omega = \frac{f\ dg - g\ df}{f^2 + g^2} \in \Omega^1(\mathbb{R}^n - Z) ,$$

where $Z = \{p \in \mathbb{R}^n \mid f(p) = 0 = g(p)\}$ is the set of common zeros of the functions f and g.

b) Let $\gamma : [0, 1] \to \mathbb{R}^n - Z$ be a smooth path such that $f(\gamma(t)) = 1$ for all $t \in [0, 1]$, and $g(\gamma(0)) = -1$ and $g(\gamma(1)) = 1$. Calculate the integral

$$\int_{[0,1]} \gamma^*(\omega) .$$