Differentiable Manifolds

TMS EXAM

September 16, 2013

Duration: 3 hr.

1. Find the tangent space to the surface $S:\ x^4 - y + z = 1$ at the point $p = (1, -1, 1)$ as a subspace of $\mathbb{R}^3$in two different ways:

(a) Using a local coordinate system at p.

(b) Exhibiting S as the preimage of a regular value under a map $f : \mathbb{R}^3 \to \mathbb{R}$ and then using the derivative of f (i.e. the induced map f_*).

2. Let $F : P^2(\mathbb{R}) \to P^1(\mathbb{R})$ be the map which is given by $F([x, y, z]) = [xy + x^2; y^2 + z^2]$. (Notation: The class of $x = (x_1, \ldots, x_{n+1})$ in $P^n(\mathbb{R})$ is denoted by $[x] = [x_1, \ldots, x_{n+1}]$.)

(a) Show that F is well defined.

(b) Choose a chart (U, ϕ) around a point $p = [x_0, y_0, z_0]$ in $P^2(\mathbb{R})$ with $y_0 \neq 0$ and a chart (V, ψ) around $F(p)$ with $F(U) \subset V$. Write the local expression of F in these charts. Is F smooth at p? Why?

(c) Compute the rank of the map F.

3. Consider the form $\omega = ydx - xdy$ in $\mathbb{R}^3$.

(a) Find the local expression of the restriction of this form to the cylinder $M : x^2 + y^2 = 1$ (i.e. $i * (\omega)$ where $i : M \to \mathbb{R}^3$ is the inclusion map) with respect to any chart of your choice.

(b) Let η be the form you have found in part (a). Find the local expression of $d\eta$ with respect to the chart you have used in part(a).

4. Let N be the unit ball in $\mathbb{R}^3$ and let f, g, h be smooth real valued functions defined on $\mathbb{R}^3$. Using Stokes Theorem write te the integral of $\omega = f dy \wedge dz + g dz \wedge dx + h dx \wedge dy$ (more precisely the integral of the restriction of this form) over the boundary of N as an integral over N. Also write it as a (iterated) Riemannian integral.

5. Prove the following

(a) If $F : N \to M$ is a one-to-one immersion and N is compact, then F is an imbedding.

(b) If $F : N \to M$ is an immersion then each $p \in N$ has a neighborhood U such that $F|U$ is an imbedding of U in M.