Differentiable Manifolds
TMS EXAM
September 16, 2013

Duration: 3 hr.

1. Find the tangent space to the surface \( S : x^4 - y + z = 1 \) at the point \( p = (1, -1, 1) \) as a subspace of \( \mathbb{R}^3 \) in two different ways:

(a) Using a local coordinate system at \( p \).

(b) Exhibiting \( S \) as the preimage of a regular value under a map \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) and then using the derivative of \( f \) (i.e. the induced map \( f_* \)).

2. Let \( F : P^2(\mathbb{R}) \rightarrow P^1(\mathbb{R}) \) be the map which is given by \( F([x,y,z]) = [xy + x^2, y^2 + z^2] \).
(Notation: The class of \( x = (x_1, \ldots, x_{n+1}) \) in \( P^n(\mathbb{R}) \) is denoted by \( [x] = [x_1, \ldots, x_{n+1}] \).)

(a) Show that \( F \) is well defined.

(b) Choose a chart \((U, \phi)\) around a point \( p = [x_0, y_0, z_0] \) in \( P^2(\mathbb{R}) \) with \( y_0 \neq 0 \) and a chart \((V, \psi)\) around \( F(p) \) with \( F(U) \subset V \). Write the local expression of \( F \) in these charts. Is \( F \) smooth at \( p \)? Why?

(c) Compute the rank of the map \( F \).

3. Consider the form \( \omega = ydx - xdy \) in \( \mathbb{R}^3 \).

(a) Find the local expression of the restriction of this form to the cylinder \( M : x^2 + y^2 = 1 \) (i.e. \( i(\omega) \) where \( i : M \rightarrow \mathbb{R}^3 \) is the inclusion map) with respect to any chart of your choice.

(b) Let \( \eta \) be the form you have found in part (a). Find the local expression of \( d\eta \) with respect to the chart you have used in part (a).

4. Let \( N \) be the unit ball in \( \mathbb{R}^3 \) and let \( f, g, h \) be smooth real valued functions defined on \( \mathbb{R}^3 \). Using Stokes' Theorem write the integral of \( \omega = f dy \wedge dz + g dx \wedge dz + h dx \wedge dy \) (more precisely the integral of the restriction of this form) over the boundary of \( N \) as an integral over \( N \). Also write it as a (iterated) Riemannian integral.

5. Prove the following

(a) If \( F : N \rightarrow M \) is a one-to-one immersion and \( N \) is compact, then \( F \) is an imbedding.

(b) If \( F : N \rightarrow M \) is an immersion then each \( p \in N \) has a neighborhood \( U \) such that \( F|_U \) is an imbedding of \( U \) in \( M \).