METU MATHEMATICS DEPARTMENT
PRELIMINARY EXAMINATION
GEOMETRY MATH 505

SEPTEMBER 17, 2014

1.) Let ω be the closed 1-form

$$\omega = \frac{x\ dy - y\ dx}{x^2 + y^2} \ \in \ \Omega^1(\mathbb{R}^2 - \{0\}).$$

a) Calculate the integral $\int_{S^1} \omega$, where S^1 is the unit circle in the plane.

b) Use Stokes' Theorem to show that the integral $\int_C \omega = 0$, where $C = \{(x, y) \mid (x-5)^2 + y^2 = 1\}$.

c) Is there a smooth map $\phi : S^1 \times [0,1] \to \mathbb{R}^2 - \{(0,0)\}$, where $\phi(S^1 \times \{0\}) = S^1$ and $\phi(S^1 \times \{1\}) = C$, so that ϕ is a diffeomorphism when restricted to each of the boundary components of the cylinder? Justify your answer!

2.) Consider the Möbius band as the following quotient manifold

$$MB = \mathbb{R} \times (-1,1)\ /(x,y) \sim (x+1, -y)\ .$$

a) Let $P : \mathbb{R} \times (-1,1) \to MB$ be the quotient map and

$$\sigma : \mathbb{R} \times (-1,1) \to \mathbb{R} \times (-1,1)$$

be the map given by $\sigma(x,y) = (x+1, -y)$. Show that for any smooth function $f : \mathbb{R} \times (-1,1) \to \mathbb{R}$ satisfying $f = -f \circ \sigma$, there is some $(x_0, y_0) \in \mathbb{R} \times (-1,1)$ with $f(x_0, y_0) = 0$.

b) Use Part (a) to show that for any 2-form ω on the Möbius band there is some $(x_0, y_0) \in \mathbb{R} \times (-1,1)$ with $\omega(P(x_0, y_0)) = 0$. Conclude that MB is not orientable.

3.) Show that the subset $\mathbb{R}^3$ given by

$$T^2 = \{(x,y,z) \in \mathbb{R}^3 \mid [(x^2 + y^2 + z^2) + 3]^2 = 16(x^2 + y^2)\}$$

is a submanifold. Show that it is diffeomorphic to the to the submanifold

$$\{(x_1, y_1, x_2, y_2) \in \mathbb{R}^4 \mid x_1^2 + y_1^2 = 1 = x_2^2 + y_2^2\}$$

via the map $F(x, y, z) = (\sqrt{x^2+y^2}-2, z, \frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}})$. Determine F^{-1}.

4.) Let $\omega = f(x,y)dx + g(x,y)dy$ be a one-form on $\mathbb{R}^2 - \{(0,0)\}$.

a) Let C_R be the circle with center at the origin and radius $R > 0$, whose parametrization is given by $x = R\cos\theta$, $y = R\sin\theta$, $0 \leq \theta \leq 2\pi$. Assume that $|f(x,y)| \leq \frac{1}{\sqrt[4]{x^2+y^2}}$ and $|g(x,y)| \leq \frac{1}{\sqrt[4]{x^2+y^2}}$, for all $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$. Show that $|\int_{C_R} \omega| \leq 4\pi\sqrt{R}$.

b) Assume that the one-form ω is also closed. Use Stokes' theorem to show that $\int_{C_R} \omega = \int_{C_1} \omega$, for all $R > 0$.

c) Show that $\int_{C_R} \omega = 0$, for all $R > 0$. Conclude that ω is an exact form.