Geometry

TMS EXAM

September 28, 2017

Duration: 3 hr.

1. Let M, N be smooth manifolds

(a) Let a be a point in N and let $i_M : M \to M \times N$ be the map given by $i_M(p) = (p, a)$. Show that i_M is smooth.

(b) Show that there is an isomorphism between the spaces $T_{(p,q)}(M \times N)$ and $T_p(M) \times T_q(N)$

(c) Show that $(i_M)_*(v) = (v, 0)$ for $v \in T_p(M)$ under the identification mentioned in part (b).

2. Let $F : P^2(\mathbb{R}) \to P^2(\mathbb{R})$ be defined by

$$F([x : y : z]) = [x^4 + z^4 : x^2y^2 : y^4]$$

(a) Show that F is well defined and smooth.

(b) Find the rank of this map at the point $[0 : 1 : 1]$.

3. (a) Let M be a smooth oriented manifold with boundary. Describe the boundary orientation on the manifold ∂M.

(b) Let $f : X \to Y$ between manifolds X and Y. If $X = \partial W$ for some compact smooth oriented manifold W, and ω is a closed n-form on Y with $n = \dim X$, then show that

$$\int_X f_*^*(\omega) = 0$$

4. Let S^3 be the unit sphere in $\mathbb{R}^4$, $i : S^3 \to \mathbb{R}^4$ the inclusion map and ω be the differential form $x^1dx^2 - x^2dx^1 + x^3dx^4 - x^4dx^3$ on $\mathbb{R}^4$.

(a) Choose any chart on S^3 and find the components of the pull-back form $i^*(\omega)$ with respect to this chart.

(b) Show that the restriction of the form ω to S^3 (i.e. the form $i^*(\omega)$) is never zero on S^3.