1. Consider \( f : \mathbb{RP}^2 \to \mathbb{R} \) defined as 
\[
 f([x : y : z]) = \frac{x^2 + 3y^2}{x^2 + y^2 + z^2}
\]

(a) Show that \( f \) is well-defined and smooth.
(b) Find the rank of \( f \) at the point \( P = [1, 1, 0] \).
(c) Prove that \( S = \{[x : y : z] | y^2 - x^2 = 2z^2 \} \) is an embedded submanifold of \( \mathbb{RP}^2 \).

2. Give an example for each of the followings. Explain your answers.

(a) \( N \subset \mathbb{R}^2 \) is a smooth manifold but not a submanifold of \( \mathbb{R}^2 \).
(b) An injective immersion of a manifold whose image is not an embedded submanifold.
(c) A non-orientable submanifold of an orientable manifold.
(d) A smooth map between manifolds such that preimage of a critical value is an embedded submanifold.

3. Let \( U = \{(x, y, z) \mid z \neq 0\} \) and let \( F : U \to V \) be defined as
\[
 F(x, y, z) = (x - \frac{y^2}{2z}, \frac{y}{z}, z)
\]

where \( V = \{(u, v, w) \mid v \neq 0\} \) Consider the vector fields
\[
 X = y \frac{\partial}{\partial x} + z \frac{\partial}{\partial y}
\]
\[
 Y = x \frac{\partial}{\partial x} - 2xy^2 \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}
\]
and 2-form \( \varphi = (w^2 - v)du \wedge dv + (uv + 3w)dv \wedge dw \)

Compute the following quantities.

(a) The push-forward \( F_* X \).
(b) The Lie bracket \([X, Y]\).
(c) \( d(F^* \varphi) \).
Problem 4. (a) Show that if $M \times \mathbb{R}$ is orientable, then $M$ is orientable. (Hence, if $M \times \mathbb{R}^n$ is orientable, then $M$ is orientable.)
(b) Show that any open submanifold of an orientable manifold is orientable.
(c) Prove that $M \times N$ is orientable if and only if $M$ and $N$ are orientable.

Problem 5. Suppose $M$ is a compact, connected and orientable $n$-manifold without boundary. Let $\theta \in \Omega^{n-1}(M)$ be any $(n - 1)$-differential form. Show that $d\theta$ must vanish at some point of $M$. 