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Department of Mathematics
TMS Exam in Geometry
November 2^{th}, 2020
Duration: 180 minutes

1. Consider $f : \mathbb{RP}^2 \longrightarrow \mathbb{R}$ defined as

$$f([x:y:z]) = \frac{x^2+3y^2}{x^2+y^2+z^2}$$

(a) Show that f is well-defined and smooth.
(b) Find the rank of f at the point $P = [1,1,0]$.
(c) Prove that $S = \{[x:y:z] \,|\, y^2 - x^2 = 2z^2\}$ is an embedded submanifold of $\mathbb{RP}^2$.

2. Give an example for each of the followings. Explain your answers.

(a) $N \subset \mathbb{R}^2$ is a smooth manifold but not a submanifold of $\mathbb{R}^2$.
(b) An injective immersion of a manifold whose image is not an embedded submanifold.
(c) A non-orientable submanifold of an orientable manifold.
(d) A smooth map between manifolds such that preimage of a critical value is an embedded submanifold.

3. Let $U = \{(x,y,z) \,|\, z \neq 0\}$ and let $F : U \longrightarrow V$ be defined as

$$F(x,y,z) = (x - \frac{y^2}{2z}, z, \frac{y}{z})$$

where $V = \{(u,v,w) \,|\, v \neq 0\}$ Consider the vector fields

$$X = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}$$
$$Y = x\frac{\partial}{\partial x} - 2xy^2\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

and 2-form $\varphi = (w^2 - v)du \wedge dv + (uv + 3w)dv \wedge dw$

Compute the following quantities.

(a) The push-forward $F_* X$.
(b) The Lie bracket $[X, Y]$.
(c) $d(F^*\varphi)$.

Problem 4. (a) Show that if $M \times \mathbb{R}$ is orientable, then M is orientable.(Hence, if $M \times \mathbb{R}^n$ is orientable, then M is orientable.)
(b) Show that any open submanifold of an orientable manifold is orientable.
(c) Prove that $M \times N$ is orientable if and only if M and N are orientable.

Problem 5. Suppose M is a compact, connected and orientable n-manifold without boundary. Let $\theta \in \Omega^{n-1}(M)$ be any $(n-1)$-differential form. Show that $d\theta$ must vanish at some point of M.