

METU-MATHEMATICS DEPARTMENT
Graduate Preliminary Examinations

Geometry

Duration: 3 hours

February 18, 2005

1. Consider the set $M = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 = 1, z^2 + w^2 = 1\} \subseteq \mathbb{R}^4$.
 - (a) Prove that M is an (imbedded) submanifold of \mathbb{R}^4 .
 - (b) Describe the tangent vectors of M at an arbitrary point $(a, b, c, d) \in M$.
 - (c) Write down a nowhere vanishing vector field on M .
 - (d) Let $\omega = (ydx - xdy) \wedge (wdz - zdw) \in \Omega(\mathbb{R}^4)$. Show that $\int_M i_*(\omega) > 0$ where $i : M \rightarrow \mathbb{R}^4$ is the inclusion map (Hint: Write a local parametrization for M).
 - (e) A consequence of Poincaré Lemma is that every closed form on \mathbb{R}^n for any n is also exact. Prove that there exists no 4-form $\theta \in \Omega(\mathbb{R}^4)$ with $d\theta = 0$ such that $\int_M i^*(\theta) \neq 0$.

2. Consider the the $(k - 1)$ dimensional sphere S^{k-1} as a submanifold of S^k via the usual embeddding $(x_1, x_2, \dots, x_k) \rightarrow (x_1, x_2, \dots, x_k, 0)$. Show that the orthogonal complement to $T_p(S^{k-1})$ in $T_p(S^k)$ is spanned by the vector $(0, 0, \dots, 1)$.

3. Let ω be a compactly supported 2-form
$$\omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$$
on \mathbb{R}^3 . Let S be the graph of a function $G : \mathbb{R}^2 \rightarrow \mathbb{R}$. Compute the integral $\int_S \omega$, and show that it is equal to $\int_{\mathbb{R}^2} (\vec{F} \cdot \vec{u}) \|\vec{n}\| dx_1 \wedge dx_2$ where $\vec{F} = (f_1, f_2, f_3)$, $\vec{u} = \frac{\vec{n}}{\|\vec{n}\|}$ with $\vec{n} = (-\frac{\partial G}{\partial x_1}, -\frac{\partial G}{\partial x_2}, 1)$.

4. Consider the sets
$$M_1 = \{[u, v, w] \in \mathbb{R}P^2 \mid u^2 + v^2 = w^2\} \subseteq \mathbb{R}P^2 .$$
$$M_2 = \{[u, v, w] \in \mathbb{R}P^2 \mid u^2 - v^2 = w^2\} \subseteq \mathbb{R}P^2 .$$

- (a) Prove that M_1 is an (imbedded) submanifold of $\mathbb{R}P^2$ diffeomorphic to \mathbf{S}^1 (Hint: Consider the image of M_1 under a suitable chart of $\mathbb{R}P^2$).
- (b) Find a diffeomorphism $F : \mathbb{R}P^2 \rightarrow \mathbb{R}P^2$ such that $F(M_1) = M_2$.