Differentiable Manifolds
TMS EXAM
11 February 2013

Duration: 3 hr.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

\[ f(x, y) = x^3 + xy + y^3 + 1 . \]

For which of the points $p = (0, 0)$, $p = (1/3, 1/3)$, $p = (-1/3, -1/3)$ is $f^{-1}(f(p))$ an imbedded submanifold in $\mathbb{R}^2$?

2. Let $M$ be the hyperboloid of two sheets given by $y^2 - z^2 - x^2 = 1$.

   (a) Let $p \in M$. Explain how we can identify $T_pM$ by a subspace of $\mathbb{R}^3$ using a chart at $p$.
   
   (b) Describe $T_p(M)$ as a subspace of $\mathbb{R}^3$ if $p = (0, 2, \sqrt{3})$.
   
   (c) Determine whether the map which assigns to each point $q = (x, y, z)$ the vector $(y, x + z, y)$ is a smooth vector field on $M$.

3. Let $F : M \to N$ be a smooth function between the manifolds $M$ and $N$ and let $a$ be a smooth function on $M$.

   (a) Show that $F^*(da) = d(F^*(a))$

   (b) Verify the formula $F^*d = dF^*$ on the forms of type $\phi_1 \wedge \phi_2$ where $\phi_1$ and $\phi_2$ are 1-forms.

   (c) Let $g : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

   \[ g(x, y, z) = (xy, x^2yz) \]

   Compute $g^*(2xydx \wedge dy)$

4. Let

\[ \alpha = \frac{1}{2\pi} \frac{x dy - y dx}{x^2 + y^2} \]

   (a) Prove that $\alpha$ is a closed 1-form on $\mathbb{R}^2 \setminus 0$

   (b) Compute the integral of $\alpha$ over the unit circle $S^1$.

   (c) How does this shows that $\alpha$ is not exact?