GEOMETRY TMS EXAM
February 17, 2016

Duration: 3 hours.

(1) Show that $N = \{[x:y:z:w] \in \mathbb{R}P^3 | x^3 + y^3 + z^3 + w^3 = 0\}$ is an embedded submanifold of $\mathbb{R}P^3$, real projective space of dimension 3, and compute its dimension.

(2) Let M be an orientable smooth manifold and fix an orientation for unit circle S^1. Given a smooth map $\gamma : S^1 \longrightarrow M$ and a differential 1-form $\alpha \in \Omega^1(M)$ define $\int_\gamma \alpha := \int_{S^1} \gamma^*(\alpha)$.

a) Show that if α is exact then for any $\gamma : S^1 \longrightarrow M$,

$$\int_\gamma \alpha = 0.$$

b) Show that if $d\alpha = 0$, and $H : [0,1] \times S^1 \longrightarrow M$ is a smooth map then,

$$\int_{\gamma_0} \alpha = \int_{\gamma_1} \alpha,$$

where $\gamma_0(\theta) = H(0, \theta)$ and $\gamma_1(\theta) = H(1, \theta)$.

(3)) Let $O(n)$ denotes the orthogonal $n \times n$ real matrices and $M(n)$ denotes $n \times n$ real matrices.

a) Show that the tangent space of $O(n)$ at the identity matrix, $T_I O(n)$ is the space of all anti-symmetric matrices.

b) Show that for any $A \in O(n)$, $T_A O(n) = \{XA | X^T = -X\}$.

c) Show that if $X \in T_I O(n)$ then $e^X \in O(n)$ where $e^X = I + X + \frac{1}{2}X^2 + \frac{1}{3!}X^3 + \cdots$.

d) Consider the smooth map $exp : M(n) \longrightarrow M(n)$, defined as $exp(X) = e^X$. Show that the differential $dexp(\mathbf{0})$ at zero matrix $\mathbf{0} \in M(n)$ is the identity linear transformation.

(4)) Let Z be the preimage of a regular value $y \in Y$ under the smooth map $F : X \longrightarrow Y$ between smooth manifolds X and Y. Prove that the kernel of the derivative $dF_x : T_x X \longrightarrow T_y Y$ at any point $x \in Z$ is precisely the tangent space to Z at x, $T_x Z$.

(5)) Define $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ by $F(u,v) = (u, v, u^2 - v^2)$. On $\mathbb{R}^2$ with coordinates (u,v) consider the following vector fields; $U_1 = u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}$ and $U_2 = u\frac{\partial}{\partial u}$ and on $\mathbb{R}^3$ with coordinates (x,y,z) consider 2-form $\omega = ydx \wedge dz + xdy \wedge dz$ and 1-form $\eta = zdx + xdy + ydz$. Compute the following:

a) $F_*[U_1, U_2]$
b) $d\omega$
c) $F^*(d\eta)$
d) $F^*(\omega)(p)[V_1, V_2]$ where $V_1 = (1,2)$ and $V_2 = (0,1)$ are the vectors in $T_p\mathbb{R}^2$, for $p = (1,1) \in \mathbb{R}^2$
e) $\omega_{F(p)}(X_1, X_2)$ where $X_1 = F_*(V_1)$ and $X_2 = F_*(V_2)$.