Duration: 3 hours.

(1) Show that \( N = \{ [z : y : x : w] \in \mathbb{R}P^3 | x^3 + y^3 + z^3 + w^3 = 0 \} \) is an embedded submanifold of \( \mathbb{R}P^3 \), real projective space of dimension 3, and compute its dimension.

(2) Let \( M \) be an orientable smooth manifold and fix an orientation for unit circle \( S^3 \). Given a smooth map \( \gamma : S^3 \to M \) and a differential 1-form \( \alpha \in \Omega^1(M) \) define \( \int_\gamma \alpha := \int_{S^3} \gamma^*(\alpha) \).

a) Show that if \( \alpha \) is exact then for any \( \gamma : S^3 \to M \),
\[
\int_\gamma \alpha = 0.
\]

b) Show that if \( d\alpha = 0 \), and \( H : [0,1] \times S^3 \to M \) is a smooth map then,
\[
\int_{\gamma_0} \alpha = \int_{\gamma_1} \alpha,
\]
where \( \gamma_0(\theta) = H(0,\theta) \) and \( \gamma_1(\theta) = H(1,\theta) \).

(3) Let \( O(n) \) denotes the orthogonal \( n \times n \) real matrices and \( M(n) \) denotes \( n \times n \) real matrices.

a) Show that the tangent space of \( O(n) \) at the identity matrix, \( T_SS(n) \) is the space of all anti-symmetric matrices.

b) Show that for any \( A \in O(n) \), \( T_SS(n) = \{ X | AX = XA \} \).

c) Show that if \( X \in T_SS(n) \) then \( e^X \in O(n) \) where \( e^X = I + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \cdots \).

d) Consider the smooth map \( \exp : M(n) \to M(n) \), defined as \( \exp(X) = e^X \). Show that the differential \( d\exp(0) \) at zero matrix \( 0 \in M(n) \) is the identity linear transformation.

(4) Let \( Z \) be the preimage of a regular value \( y \in Y \) under the smooth map \( F : X \to Y \) between smooth manifolds \( X \) and \( Y \). Prove that the kernel of the derivative \( dF_x : T_xX \to T_yY \) at any point \( x \in Z \) is precisely the tangent space to \( Z \) at \( x \).

(5) Define \( F : \mathbb{R}^2 \to \mathbb{R}^3 \) by \( F(u,v) = (u, u^2 - v^2) \). On \( \mathbb{R}^2 \) with coordinates \((u, v)\) consider the following vector fields: \( U_1 = u \frac{\partial}{\partial v} - v \frac{\partial}{\partial u} \) and \( U_2 = u \frac{\partial}{\partial u} \) and on \( \mathbb{R}^3 \) with coordinates \((x, y, z)\) consider 2-form \( \omega = y dx \wedge dz + x dy \wedge dx \) and 1-form \( \eta = x dy + y dx \). Compute the following:

a) \( F_x[U_1, U_2] \)

b) \( d\omega \)

c) \( F^*(dy) \)

d) \( F^*(\omega)([V_1, V_2]) \) where \( V_1 = (1, 2) \) and \( V_2 = (0, 1) \) are the vectors in \( T_p\mathbb{R}^2 \), for \( p = (1, 1) \in \mathbb{R}^2 \)

e) \( \omega_{F(p)}(X_1, X_2) \) where \( X_1 = F_*(V_1) \) and \( X_2 = F_*(V_2) \).