19.9.2007

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

   Using any method you like, determine the reduced and full QR-factorization of A

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2. Consider the linear system of equations $Ax = b$ with

$$A = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}.$$

a can be positive or negative.

(a) Find the range of values of a will the Jacobi method always convergent by computing the exact eigenvalues of the iteration matrix ?

(b) Find the range of values of a will the Gauss-Seidel method always convergent by estimating the eigenvalues of the iteration matrix using Gerschgorin's circle theorem?

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3. Assume that $A \in \mathbb{C}^{mxn}$ with $m > n$ has a full rank. Show how to choose ϵ so that the 2−norm condition number of

$$B = \begin{pmatrix} A \\ \epsilon^{1/2} I \end{pmatrix} \qquad (I \; nxn \text{ identity matrix})$$

is equal to the square root of the 2−norm condition number of A.

(Hint: Consider B^*B and the singular value decomposition of A).

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4. (a) Assume $A^TAx = A^Tb$ and $(A^TA+E)\hat{x} = A^Tb$ with $2||E||2 \leq \sigma_n(A)^2$. Show that if $r = b - Ax$ and $\hat{r} = b - A\hat{x}$, then

$$\hat{r} - r = (I - A(A^TA + E)^{-1}A^T)Ax$$

and

$$||\hat{r} - r||_2 \leq 2\kappa_2(A)\frac{||E||_2}{||A||_2}||x||_2.$$

(b) Assume $A^TAx = A^Tb$ and $A^TA\hat{x} = A^Tb + e$ where $||e||_2 \leq \epsilon||A^T||_2||b||_2$ and A has full column rank. Show that

$$\frac{||x - \hat{x}||_2}{||x||_2} \leq \epsilon\kappa_2(A)^2\frac{||A^2||_2||b||_2}{||A^Tb||_2}.$$
