FALL 2009

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. To produce QR factorization of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 2 & -4 & 5 \end{bmatrix}.$$

   (a) use Householder transformation,

   (b) Gram-Schmidt orthogonalization.

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2. For solving the linear system of equations $Ax = b$ you are given the iterative equations

$$x^{(k+1)} = Bx^{(k)} + c, \quad k = 0, 1, 2, \cdots$$

with an initial vector $x^{(0)}$ and a known vector c. The $n \times n$ iteration matrix B satisfies

$$\sum_{i=1}^{n} |b_{ij}| \leq M < 1, \qquad j = 1, 2, \cdots, n.$$

Show that the iteration converges to x for any starting vector $x^{(0)}$
(i.e. show that $\lim_{k \to \infty} e^{(k)} = \lim_{k \to \infty} |x^{(k)} - x| = 0$).

3. Let $Ax = b$, of order n, be uniquely solvable as $x = A^{-1}b$.

(a) If $\hat{x}$ denotes the approximate solution of the system relative to small perturbations in the right hand side b, show that

$$\frac{\| r \|}{\| A \|} \leq \| e \| \leq \| A^{-1} \| \| r \|$$

where $e = x - \hat{x}$ (the error) and $r = b - A\hat{x}$ (the residual).
$\| \cdot \|$ denotes any matrix or vector norm.

(b) If C is the computed inverse of A, define the residual matrix by

$$R = I - CA$$

and show that

$$\| e \| \leq \frac{\| Cr \|}{1 - \| R \|}$$

where $e = x - \hat{x}$, $r = b - A\hat{x}$ and again $\hat{x}$ is the approximate solution to $Ax = b$.

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4. Suppose that $Q = I + YTY^T$ is orthogonal where $Y \in R^{n\times j}$ and $T \in R^{j\times j}$ is upper triangular. Show that if $Q_+ = QP$ where $P = I - 2vv^T/v^Tv$ is a Householder matrix, then Q_+ can be expressed in the form $Q_+ = I + Y_+T_+Y_+^T$ where $Y_+ \in R^{n\times(j+1)}$ and $T_+ \in R^{(j+1)\times(j+1)}$ is upper triangular.