September, 2015

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Consider the iterative method

$$x^{(k+1)} = Ax^{(k)} + b$$

where $A = \begin{bmatrix} 5 & 3/2 \\ 4 & 4 \end{bmatrix}$ and b is an arbitrary vector.

   (a) Does this iteration converge for arbitrary initial vectors, $x^{(0)}$.

   (b) Find α, if possible, so that the following iteration converges for arbitrary initial vectors, $x^{(0)}$:

$$\begin{aligned} y^{(0)} &= x^{(0)} \\ x^{(k+1)} &= Ay^{(k)} + b \\ y^{(k+1)} &= \alpha y^{(k)} + (1-\alpha)x^{(k+1)} \end{aligned}$$

   where A is the matrix given above.

2. Show that the singular values of the following matrices are the same as their eigenvalues

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 8 & 0 \\ 1 & 0 & 8 \end{bmatrix}.$$

(Hint:Compute the singular value decomposition of the matrices A and B).

3. The quantity A is going to be computed

$$A = \frac{x^3\sqrt{y}}{z^2}$$

with the values

$$x = 8.36, \quad y = 80.46, \quad z = 25.8$$

where the absolute errors are

$$e_x = 0.01, \quad e_y = 0.02, \quad e_z = 0.03, \text{ for } x, y, \text{ and } z, \text{ respectively.}$$

   (a) Find the upper bound for the relative error Rel_A of A.

   (b) Find the absolute error e_A in A.

4. Consider the $n \times n$, nonsingular matrix A. The Frobenious norm of A is given by

$$\|A\| = \Big(\sum_{i,j} |a_{i,j}|^2\Big)^{1/2}$$

(a) Construct the perturbation δA, with smallest Frobenious norm such that $A - \delta A$ is singular. (Hint: you might use of the primary decompositions of A)

(b) What is the Frobenious norm of this special δA?

(c) Prove that it is the smallest such perturbation?