September, 2025

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Let $A = [a_{ij}]$ be a square, nonsingular matrix of size n with nonzero diagonal entries, and x and b are n-vectors. Consider the numerical solution of the linear system $Ax = b$ by an iterative scheme of the form
$$x^{(k+1)} = Bx^{(k)} + c$$
with appropriate sizes of B and c.

   (a) Find the iteration matrix of the Jacobi iterations and the Gauss-Seidel iterations. That is, find B and c specifically for both methods.

   (b) Let $n = 2$. Show that $\rho(B_J) = \sqrt{\left|\dfrac{a_{21}a_{12}}{a_{11}a_{22}}\right|}$ for the Jacobi iterations and $\rho(B_{GS}) = \left|\dfrac{a_{21}a_{12}}{a_{11}a_{22}}\right|$ for the Gauss-Seidel iterations, where $\rho(B_J)$ and $\rho(B_{GS})$ are the spectral radii of the Jacobi and Gauss-Seidel iterations, respectively.

   (c) Let $n = 2$. Prove that the Jacobi iterations converge if and only if Gauss-Seidel iterations converge.

2. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a diagonalizable matrix with eigenvalues
$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n| > 0,$$
and corresponding linearly independent eigenvectors $v_1, v_2, \cdots, v_n$, such that
$$\mathbf{A}v_i = \lambda_i v_i, \quad \textbf{for} \quad i = 1, 2, \cdots, n.$$

   (a) Give the condition(s) required for convergence of the Power Method to the dominant eigenvector. Write a simple pseudocode for the Power Method to approximate the eigenvector associated with the largest magnitude eigenvalue of $\mathbf{A}$.

   (b) Assume that the condition(s) in part (a) are satisfied. Show that the Power Method converges to the dominant eigenvector. (Assume that $\lambda_1 > 0$).

   (c) Using your solution in part (b), show that the Rayleigh Quotient converges to the dominant eigenvalue.

   (d) Suppose $|\lambda_{n-1}| > |\lambda_n|$. Modify the Power Method to approximate an eigenvector corresponding to the smallest magnitude eigenvalue. Explain why your modification works. Give the condition(s) required for convergence and write a simple pseudocode for this procedure.

3. Let $A \in \mathbb{C}^{n\times n}$ and assume there exists $p \geq 1$ such that $\|A\|_p < 1$, where $\|\cdot\|_p$ is a vector-induced matrix norm. Prove the followings:

   (a) $I - A$ is invertible.

   (b) $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$.

   (c) $\|A\|_q \, \|A^{-1}\|_q \geq 1$, for all $1 \leq q \leq \infty$.

   (d) $\dfrac{1}{1+\|A\|_p} \leq \|(I-A)^{-1}\|_p \leq \dfrac{1}{1-\|A\|_p}$.