5.2.2007

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Calculate the QR factorization of the matrix

$$A = \begin{pmatrix} \sqrt{2} & 0 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

   (a) using Given's rotation,

   (b) Householder's reflection.

(c) Let A be a symmetric matrix and let λ and x be an eigenvalue-eigenvector pair for$A$ with $||x||_2 = 1$. Let P be an orthogonal matrix for which $Px = e_1 = [1, 0, \cdots, 0]^T$ and consider the similar matrix $B = PAP^T$.

i. Show that the first row and column of B are zero except for the diagonal element which equals λ.

ii. For the matrix

$$A = \begin{bmatrix} 2 & 10 & 2 \\ 10 & 5 & -8 \\ 2 & -8 & 11 \end{bmatrix}$$

with $\lambda = 9$ as an eigenvalue with associated eigenvector $x = [\frac{2}{3}, \frac{1}{3}, \frac{2}{3}]^T$. Produce a Householder matrix P for which $Px = e_1$ and then produce $B = PAP^T$.

iii. By using the matrix B in part (b) find two other eigenvalues of A.

(d) Let $A \in \mathbf{R}^{n \times n}$ be a symmetric positive definite matrix, $b \in \mathbf{R}^n$, $x^* := A^{-1}b$ and assume that $x \in \mathbf{R}^n$ is some approximation of x^* with $r := b - Ax$ denoting the associated residual.

i. Show that

$$a_2(r) \leq \| x - x^* \|_2 \leq C_2(A) a_2(r),$$

where

$$a_2(r) := \frac{\| r \|_2^2}{\| A^T r \|_2}, \; C_2(A) := \sup_{y \neq 0} \frac{\| A^T y \|_2 \| A^{-1} y \|_2}{\| y \|_2^2}.$$

ii. Prove the estimate

$$C_2(A) \leq \frac{1}{2}(\kappa_2(A) + \kappa_2^{-1}(A)),$$

where $\kappa_2(A) := \| A^{-1} \|_2 \| A \|_2$.

[Hint: Use the Kantorovich inequality]

$$< By, y >< B^{-1}y, y > \leq \frac{(\lambda_{\min}(B) + \lambda_{\max}(B))^2}{4\lambda_{\min}(B)\lambda_{\max}(B)} \| y \|_2^4$$

for symmetric positive definite matrices $B \in \mathbf{R}^{n \times n}$.]

(e) Consider the linear system of equations $Ax = b$ where $b \in \mathbf{R}^n$ and A is an $n \times n$ matrix given by

$$A = \begin{bmatrix} 4 & 1 & \cdots & \cdots & \cdots \\ 1 & 4 & 1 & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots & \cdots \\ \cdots & \cdots & 1 & 4 & 1 \\ \cdots & \cdots & \cdots & 1 & 4 \end{bmatrix},$$

i. Write down the Jacobi iteration in matrix form for this system.
ii. Find the rate of convergence of the Jacobi iteration.
iii. How many iterations are required to reduce the 2-norm error by a factor of 10^{-6}?

Hint: Eigenvalues of a $n \times n$ tridiagonal matrix

$$\begin{bmatrix} a & b & \cdots & \cdots & \cdots \\ c & a & b & \cdots & \cdots \\ \ddots & \ddots & \ddots & \ddots & \cdots \\ \cdots & \cdots & c & a & b \\ \cdots & \cdots & \cdots & a & b \end{bmatrix},$$

are given as

$$\lambda_j = a + 2\sqrt{bc}\cos\frac{j\pi}{n+1}, \quad j = 1, \cdots, n$$