1- Consider the problem of solving a nonsingular linear system $Ax = b$.

(a) If the entries of $A$ are given with perturbations (error) $\delta A$ such that

$$(A + \delta A)(x + \delta x) = b$$

where $\delta x$ is the perturbation in the solution vector $x$ due to the perturbations in the matrix $A$, prove that

$$||\delta x|| \leq \frac{\epsilon}{1 - \epsilon} ||x||$$

where $\epsilon = ||\delta A|| ||A^{-1}|| < 1$ and $|| \cdot ||$ is any matrix or vector norm.

(b) If the right hand side vector $b$ is given with perturbation $\delta b$ as

$$A(x + \delta x) = b + \delta b$$

where $\delta x$ now, is the perturbation in the solution $x$ due to the perturbation in $b$, prove that

$$\frac{||\delta b||}{||A||} \leq ||\delta x|| \leq ||A^{-1}|| ||\delta b||.$$

2- Let $A$ and $B$ are matrices with size $n \times n$ and $A$ is nonsingular. Consider solving the linear system

$$Az_1 + Bz_2 = b_1$$

$$Bz_1 + Az_2 = b_2$$

with $z_1, z_2, b_1, b_2 \in \mathbb{R}^n$ and $b_1, b_2$ are given. Find the condition for convergence of the iteration method

$$Az_1^{(m+1)} = b_1 - Bz_2^{(m)}$$

$$m \geq 0$$

$$Az_2^{(m+1)} = b_2 - Bz_1^{(m)}$$

where $m$ is the iteration number.
3- Consider the two-step Newton method for

\[ y_k = x_k - \frac{f(x_k)}{f'(x_k)}, \quad x_{k+1} = y_k - \frac{f(y_k)}{f'(y_k)} \]

where \( k \) is the iteration number \((k \geq 0)\). Show that, the iteration converges \textbf{cubically} to a simple root \( \zeta \) of \( f(x) = 0 \).

4- Let

\[ A = \begin{bmatrix}
6 & 2 & 1 \\
1 & -5 & 0 \\
2 & 1 & 4
\end{bmatrix} \]

(a) Using Gershgorin’s Theorem show that the eigenvalues of \( A \) satisfies the inequality \( 1 \leq |\lambda| \leq 9 \).

(b) Using Gershgorin’s Theorem prove that all eigenvalues of a diagonally dominant matrix are non-zero.

(c) Using \textbf{power method} with \( x_0 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T \) compute the absolutely largest eigenvalue and corresponding eigenvector approximately (perform 3 iterations).