September, 2011

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Consider the following single shift QR algorithm applied to a matrix $A \in \mathbf{R}^{\mathbf{m}\times\mathbf{m}}$:

$$\begin{aligned}
&T^{(0)} = A \\
&for \quad k = 1, 2, \ldots \\
&T^{(k-1)} - \sigma^{(k-1)} I = Q^{(k)} R^{(k)} \qquad (QR \quad \text{decomposition}) \\
&T^{(k)} = R^{(k)} Q^{(k)} + \sigma^{(k-1)} I \\
&end(k)
\end{aligned}$$

where the shifts are given by $\sigma^{(k-1)} = (T^{(k-1)})_{m,m}$ for all $k \geq 1$. Assume that none of the shifts correspond to an eigenvalue of the matrix A and the diagonal coefficients of the matrices $R^{(k)}$ produced during the decomposition of $T^{(k-1)} - \sigma^{(k-1)} I$ are nonnegative. For $k \geq 1$ define the matrices

$$Q_k = Q^{(1)} Q^{(2)} \ldots Q^{(k)} \quad \text{and} \quad R_k = R^{(k)} R^{(k-1)} \ldots R^{(1)}.$$

(a) Show that the sequence of matrices $T^{(k)}$ generated by single shift QR algorithm is orthogonally similar to the initial matrix A.

(b) Show that

$$\prod_{j=0}^{k-1} (A - \sigma^{(j)} I) = Q_k R_k \quad \text{for} \quad k \geq 1.$$

(c) Verify that the first column of Q_k is

$$Q_k e_1 = \frac{1}{(R_k)_{1,1}} \Big[\prod_{j=0}^{k-1} (A - \sigma^{(j)} I) \Big] e_1$$

That is $Q_k e_1$ is essentially the vector obtained by using a shifted power iteration starting with e_1.

(d) Suppose A is symmetric. Verify that the last column of Q_k is

$$Q_k e_m = (R_k)_{m,m} \Big[\prod_{j=0}^{k-1} (A - \sigma^{(j)} I) \Big]^{-1} e_m.$$

That is $Q_k e_m$ is essentially the vector obtained by using a shifted inverse power iteration starting with e_m.

2. Let A be an $m \times n$ matrix $(m \geq n)$.

   (a) Define the concept of a full QR factorization of A and the concept of a reduced QR factorization of A.

   (b) Describe the method of Householder triangulations for computing a reduced QR factorization of A.

   (c) Let A be an matrix $n \times n$ and let a_j be its j^{th}column. Use a QR factorization of A to show that

$$|detA| \leq \prod_{j=1}^{n} \|a_j\|_2.$$

3. Let $A \in \mathbf{R}^{\mathbf{n}\times\mathbf{n}}$ be symmetric positive definite. The iteration for solving the system $Ax = b$ is defined by

$$x^{k+1} = x^k - \alpha(Ax^k - b) \quad k \geq 0.$$

   Assume that the eigenvalues of A satisfy $0 < \lambda_1 \leq \lambda_2 \ldots \leq \lambda_n$.

   (a) Show that the iteration converges if and only if $0 < \alpha < \dfrac{2}{\lambda_n}$.

   (b) Show that the optimal choice of α is $\dfrac{2}{\lambda_1 + \lambda_n}$.

   (*You might use 2-norm*).

4. Let $A \in \mathbf{C}^{\mathbf{m}\times\mathbf{n}}$ with $m > n$.

   (a) Show that $A^*A + \epsilon I$ is Hermitian and positive definite for every positive value of ϵ.

   (b) Assume A has full rank. Show how to choose ϵ so that the 2– norm condition number of

$$B = \begin{pmatrix} A \\ \epsilon^{1/2} I \end{pmatrix},$$

   where I is a $n \times n$ identity matrix, is equal to the square root of the 2–norm condition number of A.

   (c) Use the reduced SVD of A to express the solution x to the problem

$$\min \left\| \begin{pmatrix} A \\ \epsilon^{1/2} I \end{pmatrix} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\|_2,$$

   where b is a $m \times 1$ vector.

GOOD LUCK!!!