February, 2012

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Let A and B have order n matrices with A nonsingular. Consider solving the linear system

$$\begin{aligned} Ax_1 + Bx_2 &= b_1 \\ Bx_1 + Ax_2 &= b_2 \end{aligned}$$

with x_1, x_2, b_1 and $b_2 \in \mathbb{R}^n$.

   (a) Find necessary and sufficient conditions for convergence of the iteration method for $m \geq 0$

$$\begin{aligned} Ax_1^{(m+1)} &= b_1 - Bx_2^{(m)} \\ Ax_2^{(m+1)} &= b_2 - Bx_1^{(m)} \end{aligned}$$

   (b) Repeat part (a) for the iteration method for $m \geq 0$

$$\begin{aligned} Ax_1^{(m+1)} &= b_1 - Bx_2^{(m)} \\ Ax_2^{(m+1)} &= b_2 - Bx_1^{(m+1)} \end{aligned}$$

   (c) Compare the convergence rates of the two methods given in part (a) and part (b)

2. Consider the least square problems of minimizing

$$\rho^2(x) = \|b - Ax\|^2.$$

Here A is $m \times n$ matrix of rank n, $(m \geq n)$ and $\|\cdot\|$ is the Euclidean vector norm. Let

$$A = (Q_1 \quad Q_2) \begin{pmatrix} R \\ 0 \end{pmatrix}$$

be the QR decomposition of A where Q_1, Q_2 and R are respectively, $m \times n, m \times (m-n)$, and $n \times n$.

   (a) Show that the solution of the least squares problem satisfies the QR equation $Rx = Q_1^T b$ and that the solution is unique. Further show that $\rho(x) = \|Q_2^T b\|$.
   (b) Use the QR equation to show that the least square solution satisfies the normal equations $(A^T A)x = A^T b$.

3. Given the vector $x = (0 \quad 0 \quad 0 \quad 3 \quad 4)^T$, by using Householder transformation, calculate a matrix Q for which Qx will have zeroes in its ***last two*** positions.

4. Let A be a nonsingular matrix whose leading principal submatrices are all nonsingular. Partition A as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is , say $k \times k$.

   (a) Is A_{11} singular or nonsingular? Explain.

   (b) Show that there is exactly one matrix M such that

$$\begin{pmatrix} I_{k\times k} & 0 \\ -M & I_{(n-k)\times(n-k)} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix}$$

   (c) Give the explicit formulas for M and $\tilde{A}_{22}$

   (d) Show that $\tilde{A}_{22}$ is nonsingular.

   (e) Let $A_{11} = L_1U_1$ and $\tilde{A}_{22} = L_2U_2$ be the LU factorization of A_{11} and $\tilde{A}_{22}$, respectively. Find matrices L_{12} and U_{12} such that LU decomposition of A is

$$A = \begin{pmatrix} L_1 & 0 \\ L_{12} & L_2 \end{pmatrix} \begin{pmatrix} U_1 & U_{12} \\ 0 & U_2 \end{pmatrix}$$