February, 2015

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Consider the matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find the Jacobi and Gauss-Seidel iteration matrices and their eigenvalues λ_i and μ_i, respectively. Show that $\mu_{max} = \lambda_{max}^2$

2. Consider the real system of linear equations

$$Ax = b \tag{1}$$

where A is non singular and satisfies

$$(v, Av) > 0$$

for all real v, where the Euclidean inner product is used here.

(a) Show that $(v, Av) = (v, Mv)$ for all real v where $M = \frac{1}{2}(A + A^T)$ is symmetric part of A.

(b) Prove that

$$\frac{(v, Av)}{(v, v)} \geq \lambda_{min}(M) > 0$$

where $\lambda_{min}(M)$ is the minimum eigenvalue of M.

3. An overdetermined system $Ax = b$, $(m > n)$ is written as

$$\begin{bmatrix} R \\ 0 \end{bmatrix} x \approx \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where $A \in \mathbb{R}^{m\times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $R \in \mathbb{R}^{n\times n}$, $b_1 \in \mathbb{R}^n$, $b_2 \in \mathbb{R}^{(m-n)}$, and $0 \in \mathbb{R}^{(m-n)\times n}$. Show that the least square solution x can be obtained from

$$Rx = b_1$$

and the residual vector $r = b - Ax$ satisfies

$$\|r\|_2^2 = \|b_2\|_2^2 \, .$$

4. (a) Describe the singular value decomposition (SVD) of the matrix $A \in \mathbb{C}^{m\times n}$. Include an explanation of the rank of A and how the SVD relates to the four fundamental subspaces

R(A) Range of A, R(A*) Range of A*

N(A) Nullspace of A, N(A*) Nullspace of A*

(b) Perform SVD on the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 2 & -1 \\ 1 & 0 \end{pmatrix}$$

(c) Compute the pseudo-inverse of A (the Moore-Penrose pseudo-inverse). Leave in factored form.