February, 2017

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. (a) Show that the following matrix formula (where $q \in \mathbb{R}$) can be used to calculate $\mathbf{A^{-1}}$ when the process

$$x^{(n+1)} = x^{(n)} + q(\mathbf{A}x^{(n)} - \mathbf{I})$$

converges.

(b) When $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, give the values of q for which the process in (a) can be used. Which q yields the fastest convergence ?

(c) Let $\mathbf{A}$ be a symmetric and positive definite $n \times n$ matrix with smallest eigenvalue λ_1, and greatest eigenvalue λ_2. Find q to get as fast convergence as possible ?

[Hint: Fastest convergence is obtained when the spectral radius is minimized].

2. Given an $\mathbb{R}^{n\times m}$ matrix $\mathbf{A}$ with singular value decomposition $\mathbf{A} = \mathbf{UDV^T}$. Let $\mathbf{A}^\dagger$ be the pseudo-inverse matrix of $\mathbf{A}$.

(a) Verify that the singular value decomposition of $\mathbf{A}^\dagger$ is $\mathbf{A}^\dagger = \mathbf{VD^+U^T}$, where $\mathbf{D}^+$ is the transpose of $\mathbf{D}$ with every non-zero entry replaced by its reciprocal.

(b) If $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$, find the SVD of $\mathbf{A}$.

(c) For the matrix $\mathbf{A}$ in part b), calculate $\mathbf{A}^\dagger$ through the SVD of $\mathbf{A}^\dagger$.

3. Let $\mathbf{x}$ be the solution of $A\mathbf{x} = \mathbf{b}$, where A is square and invertible. Carry out the perturbation analysis when *both* the matrix A and the vector $\mathbf{b}$ is perturbed. Let $\tilde{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x}$ such that $(A + \delta A)\tilde{\mathbf{x}} = \mathbf{b} + \delta\mathbf{b}$. Prove the following estimate:

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \le \frac{\kappa(A)}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}} \left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta\mathbf{b}\|}{\|\mathbf{b}\|} \right),$$

provided that δA is sufficiently small, in our case assume that $\|A^{-1}\| \cdot \|\delta A\| < 1$. The matrix norm is the induced norm obtained from the vector norm used and $\kappa(A) = \|A\| \cdot \|A^{-1}\|$.

4. Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix where eigenvalues are given by $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$. Suppose that $\beta \in \mathbb{R}$ and the vector $x \in \mathbb{C}^n$, $x \neq 0$ are such that $d = Ax - \beta x$. Then,

   (a) Show that

   $$\min_{1\leq\mu\leq n} |\beta - \lambda_\mu| \leq \frac{\|d\|_2}{\|x\|_2}.$$

   [Hint: Since A is Hermitian, the corresponding eigenvectors $x_1, x_2, \cdots, x_n$ form an orthonormal basis for $\mathbb{C}^n$].

   (b) Apply this result to the matrix

   $$A = \begin{bmatrix} 6 & 4 & 3 \\ 4 & 6 & 3 \\ 3 & 3 & 7 \end{bmatrix}$$

   with $\beta = 12$ and $x = (0.9, 1, 1.1)^T$ where A has eigenvalues $\lambda_1 = 13$, $\lambda_2 = 4$ and $\lambda_3 = 2$.