1. (a) Show that the following matrix formula (where \( q \in \mathbb{R} \)) can be used to calculate \( A^{-1} \) when the process
\[
x^{(n+1)} = x^{(n)} + q(Ax^{(n)} - I)
\]
converges.
(b) When \( A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \), give the values of \( q \) for which the process in (a) can be used. Which \( q \) yields the fastest convergence?
(c) Let \( A \) be a symmetric and positive definite \( n \times n \) matrix with smallest eigenvalue \( \lambda_1 \), and greatest eigenvalue \( \lambda_2 \). Find \( q \) to get as fast convergence as possible?
[Hint: Fastest convergence is obtained when the spectral radius is minimized].

2. Given an \( \mathbb{R}^{n \times m} \) matrix \( A \) with singular value decomposition \( A = UDV^T \). Let \( A^\dagger \) be the pseudo-inverse matrix of \( A \).
   (a) Verify that the singular value decomposition of \( A^\dagger \) is \( A^\dagger = VD^+U^T \), where \( D^+ \) is the transpose of \( D \) with every non-zero entry replaced by its reciprocal.
   (b) If \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} \), find the SVD of \( A \).
   (c) For the matrix \( A \) in part b), calculate \( A^\dagger \) through the SVD of \( A^\dagger \).

3. Let \( x \) be the solution of \( Ax = b \), where \( A \) is square and invertible. Carry out the perturbation analysis when both the matrix \( A \) and the vector \( b \) is perturbed. Let \( \tilde{x} = x + \delta x \) such that \( (A + \delta A)\tilde{x} = b + \delta b \). Prove the following estimate:
\[
\frac{\|\delta x\|}{\|x\|} \leq \frac{\kappa(A)}{1 - \kappa(A) \frac{\|\delta A\|}{\|A\|}} \left( \frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|} \right),
\]
provided that \( \delta A \) is sufficiently small, in our case assume that \( \|A^{-1}\| : \|\delta A\| < 1 \). The matrix norm is the induced norm obtained from the vector norm used and \( \kappa(A) = \|A\| \cdot \|A^{-1}\| \).
4. Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix where eigenvalues are given by $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$. Suppose that $\beta \in \mathbb{R}$ and the vector $x \in \mathbb{C}^n$, $x \neq 0$ are such that $d = Ax - \beta x$. Then,

(a) Show that

$$\min_{1 \leq \mu \leq n} |\beta - \lambda_\mu| \leq \frac{\|d\|_2}{\|x\|_2}.$$  

[Hint: Since $A$ is Hermitian, the corresponding eigenvectors $x_1, x_2, \cdots, x_n$ form an orthonormal basis for $\mathbb{C}^n$].

(b) Apply this result to the matrix

$$A = \begin{bmatrix} 6 & 4 & 3 \\ 4 & 6 & 3 \\ 3 & 3 & 7 \end{bmatrix}$$

with $\beta = 12$ and $x = (0.9, 1, 1.1)^T$ where $A$ has eigenvalues $\lambda_1 = 13$, $\lambda_2 = 4$ and $\lambda_3 = 2$. 