1. (a) Fit a parabola of the form \( y = a + bx^2 \) to the following data by using the least squares approximation.

\[
\begin{array}{c|cccc}
  x & 0 & 1 & -2 & 3 \\
  y & 1 & 1 & -1 & 0 \\
\end{array}
\]

(b) Consider the matrix \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \). Find the singular value decomposition (SVD) of \( A \) and the rank of \( A \).

(c) Consider the linear system \( Ax = b \), for \( b \in \mathbb{R}^3 \) and \( A \) is the matrix given in part (b). State the condition such that the equation \( Ax = b \) has a solution, and the condition such that the solution is unique.

(d) Find the pseudoinverse of the matrix \( A \) given in part (b).

(e) Find the solution of the system given in part (c) for \( b = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} \).

2. (a) Prove the identity

\[ A^{-1} - B^{-1} = A^{-1} (B - A) B^{-1} \]

and hence deduce that

\[ \| A^{-1} - B^{-1} \| \leq \| A^{-1} \| \| B - A \| \| B^{-1} \| \]

(b) Prove that if \( B = A + \delta A \) where \( \| \delta A \| \| B^{-1} \| = \delta < 1 \), then it follows that

\[ \| A^{-1} \| \leq \frac{1}{1 - \delta} \| B^{-1} \|, \quad \| A^{-1} - B^{-1} \| \leq \frac{\delta}{1 - \delta} \| B^{-1} \| \]

(c) Prove that if \( x = A^{-1} b \) and \( x + \delta x = (A + \delta A)^{-1} b \), then

\[ \| \delta x \| \leq \frac{\delta}{1 - \delta} \| x + \delta x \| \quad \text{where} \quad \delta = \| \delta A \| \| B^{-1} \| < 1 \]

and

\[ \| \delta x \| \leq \frac{\epsilon}{1 - \epsilon} \| x \| \quad \text{where} \quad \epsilon = \| \delta A \| \| A^{-1} \| < 1. \]
3. Let $A \in \mathbb{R}^{n \times n}$ be symmetric and positive definite matrix. Consider the following iteration

Choose $A_0 = A$
for $k = 0, 1, 2, \ldots$,
Compute Cholesky factor $L_k$ of $A_k$ (so $A_k = L_k L_k^T$)
Set $A_{k+1} = L_k^T L_k$
end

where $L_k$ is lower triangular with positive diagonal elements.

(a) Show that $A_k$ is similar to $A$ and that $A_k$ is symmetric positive definite (thus the iteration is well-defined).

(b) Consider

$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad a \geq c$

For this matrix, perform one step of algorithm above and write down $A_1$.

(c) Use the result from (b) to argue that $A_k$ converges to the diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, where the eigenvalues of $A$ are ordered as $\lambda_1 \geq \lambda_2 > 0$.

4. Given the matrix

$A = \begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix}$

(a) Find the reduced QR-factorization by applying Gram Schmidt orthogonalization to the columns of $A$.

(b) Find the full QR-factorization of $A$?