21.9.2007

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. Given f_i and f_i' at the points x_i, $i = 1, 2$.

   (a) Using Newton's divided difference formula, determine the cubic $P(x)$ such that
   $$P(x_i) = f_i, \qquad \text{and} \quad \frac{d}{dx}P(x_i) = f_i'.$$

   (b) Show that
   $$\int_{x_1}^{x_2} P(x)dx = (x_2 - x_1)\frac{f_1 + f_2}{2} + \frac{(x_2 - x_1)^2}{12}(f_1' - f_2').$$

   (c) What is the numerical use of formula such as that in part (b).

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2. Let $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x), \cdots$, be a sequence of orthogonal polynomials on an interval $[a, b]$ with respect to a positive weight funciton $w(x)$. Let $x_1, \cdots, x_n$ be the n zeros of $\phi_n(x)$; it is known that these roots are real and $a < x_1 < \cdots < x_n < b$.

   (a) Show that the Lagrange polynomials of degree $n - 1$,

   $$L_j(x) = \prod_{k=1, k\neq j}^{n} \frac{(x - x_k)}{x_j - x_k}, \quad 1 \leq j \leq n$$

   for these points are orthogonal to each other, i.e.,

   $$\int_a^b w(x) L_j(x) L_k(x) dx = 0, \quad j \neq k.$$

   (b) For a given function $f(x)$, let $y_k = f(x_k)$, $k = 1, \cdots, n$. Show that the polynomial $p_{n-1}(x)$ of degree at most $n - 1$ which interpolates the function $f(x)$ at the zeros $x_1, \cdots, x_n$ of the orthogonal polynomial $\phi_n(x)$ satisfies

   $$\| p_{n-1} \|^2 = \sum_{k=1}^{n} y_k^2 \| L_k \|^2$$

   in the weighted least squares norm. This norm is defined as follows: for any general function $g(x)$,

   $$\| g \|^2 = \int_a^b w(x) [g(x)]^2 dx.$$

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3. Let $f(x) = x - e^{-x}$.

(a) Prove that $f(x) = 0$ has a root $r \in (0, 1)$.

(b) Let (x_n) be the Newton's sequence related to $f(x) = 0$ with $x_0 \geq 0$. Prove that

$$0 \leq x_{n+1} \leq 1 + \frac{x_0}{2^{n+1}}, \quad \text{for all} \quad n$$

(c) Take $x_0 = 10^{10}$. How many iterations are needed to have $x_n \leq \frac{3}{2}$? Set $e_n = x_n - r$. Why for such n we have $|e_n| \leq \frac{3}{2}$?

(d) Knowing that $e_{n+1} = \frac{e_n^2 f''(\theta_n)}{2f'(x_n)}$ with θ_n between x_n and r prove that

$$|e_{n+1}| \leq 2 \left(\frac{e_0}{2}\right)^{2^{n+1}}.$$

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4. (a) Let us consider $f(x) = \alpha e^{-x}(1 + x^2)^{1/2}$ in $\Omega = [0, 1]$. For which values of α has $f(x)$ a unique fixed point in Ω.

   (b) Apply Newton's method to the function $f(x) = 1/x - a$ to find $g(x)$ such that the iterates

$$x_{k+1} = g(x_k)$$

   converge to $1/a$. Show that this iteration formula can be written in the interesting form

$$x_{k+1} f(x_{k+1}) = (x_k f(x_k))^2.$$
