1. Given $f_i$ and $f'_{i}$ at the points $x_i$, $i = 1, 2$.

   (a) Using Newton’s divided difference formula, determine the cubic $P(x)$ such that
   $$P(x_i) = f_i, \quad \text{and} \quad \frac{d}{dx}P(x_i) = f'_{i}.$$ 

   (b) Show that
   $$\int_{x_1}^{x_2} P(x)dx = (x_2 - x_1) \frac{f_1 + f_2}{2} + \frac{(x_2 - x_1)^2}{12} (f'_1 - f'_2).$$

   (c) What is the numerical use of formula such as that in part (b).
2. Let $\phi_0(x), \phi_1(x), \phi_2(x), \ldots$, be a sequence of orthogonal polynomials on an interval $[a, b]$ with respect to a positive weight function $w(x)$. Let $x_1, \ldots, x_n$ be the $n$ zeros of $\phi_n(x)$; it is known that these roots are real and $a < x_1 < \cdots < x_n < b$.

(a) Show that the Lagrange polynomials of degree $n - 1$,

$$L_j(x) = \prod_{k=1, k \neq j}^{n} \frac{(x - x_k)}{x_j - x_k}, \quad 1 \leq j \leq n$$

for these points are orthogonal to each other, i.e.,

$$\int_a^b w(x)L_j(x)L_k(x)dx = 0, \quad j \neq k.$$

(b) For a given function $f(x)$, let $y_k = f(x_k), \ k = 1, \ldots, n$. Show that the polynomial $p_{n-1}(x)$ of degree at most $n - 1$ which interpolates the function $f(x)$ at the zeros $x_1, \ldots, x_n$ of the orthogonal polynomial $\phi_n(x)$ satisfies

$$\| p_{n-1} \|^2 = \sum_{k=1}^{n} y_k^2 \| L_k \|^2$$

in the weighted least squares norm. This norm is defined as follows: for any general function $g(x)$,

$$\| g \|^2 = \int_a^b w(x)[g(x)]^2 dx.$$
3. Let \( f(x) = x - e^{-x} \).

(a) Prove that \( f(x) = 0 \) has a root \( r \in (0, 1) \).

(b) Let \((x_n)\) be the Newton’s sequence related to \( f(x) = 0 \) with \( x_0 \geq 0 \). Prove that

\[
0 \leq x_{n+1} \leq 1 + \frac{x_0}{2^{n+1}}, \quad \text{for all } n
\]

(c) Take \( x_0 = 10^{10} \). How many iterations are needed to have \( x_n \leq \frac{3}{2} \)? Set \( e_n = x_n - r \). Why for such \( n \) we have \( |e_n| \leq \frac{3}{2} \)?

(d) Knowing that \( e_{n+1} = \frac{e_n f'(\theta_n)}{f'(x_n)} \) with \( \theta_n \) between \( x_n \) and \( r \) prove that

\[
|e_{n+1}| \leq 2 \left( \frac{e_0}{2} \right)^{2^{n+1}}.
\]
4. (a) Let us consider \( f(x) = \alpha e^{-x}(1 + x^2)^{1/2} \) in \( \Omega = [0, 1] \). For which values of \( \alpha \) has \( f(x) \) a unique fixed point in \( \Omega \).

(b) Apply Newton’s method to the function \( f(x) = 1/x - a \) to find \( g(x) \) such that the iterates

\[
x_{k+1} = g(x_k)
\]

converge to \( 1/a \). Show that this iteration formula can be written in the interesting form

\[
x_{k+1}f(x_{k+1}) = (x_kf(x_k))^2.
\]