September, 2009

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. By using Newton form of an interpolating polynomial show that

   (a) If $p(x) \in P_n$ interpolates a function f at a set of $n+1$ distinct nodes $x_0, x_1, \cdots, x_n$ and if t is a point different from the nodes, then

   $$f(t) - p(t) = f[x_0, x_1, \cdots, x_n, t] \prod_{j=0}^{n} (t - x_j).$$

   (b) If $f \in C^n[a,b]$ and if $x_0, x_1, \cdots, x_n$ are distinct points in $[a,b]$ then there exists a point $\xi \in (a,b)$ such that

   $$f[x_0, x_1, \cdots, x_n] = \frac{f^{(n)}(\xi)}{n!}.$$

   (c) If f is a polynomial of degree k, then for $n > k$

   $$f[x_0, x_1, \cdots, x_n] = 0.$$

   Note: P_n is the set of all n-th degree polynomials, $\prod$ denotes product notation $f[x_0, x_1, \cdots, x_n]$ is the n-th order divided difference of f, $f^{(n)}$ denotes n-th derivative of f.

2. Let $\phi_0(x), \phi_1(x), \phi_2(x), \cdots,$ be a sequence of orthogonal polynomials ($\phi_j(x)$ is a jth degree polynomial) on an interval $[a, b]$ with respect to a positive weight function $w(x)$. Let $x_1, \cdots, x_n$ be the n zeros of $\phi_n(x)$; it is known that these roots are real and

$$a < x_1 < \cdots, < x_n < b.$$

(a) Show that the Lagrange polynomials of degree $n - 1$,

$$L_j(x) = \prod_{\substack{k= \\ k \neq j}} \frac{(x - x_k)}{(x_j - x_k)}, \quad 1 \leq < \leq n$$

for these points are orthogonal to each other, i.e.,

$$\int_a^b w(x) L_j(x) L_k(x) dx = 0, \quad j \neq k.$$

(b) For a given function $f(x)$, let $y_k = f(x_k)$, $k = 1, \cdots, n$. Show that the polynomial $p_{n-1}(x)$ of degree at most $n-1$ which interpolates the function $f(x)$ at the zeros $x_1, \cdots, x_n$ of the orthogonal polynomial $\phi_n(x)$ satisfies

$$||p_{n-1}||^2 = \sum_{k=1}^{n} y_k^2 \, ||L_k||^2$$

in the weighted least squares norm. This norm is defined as follows: for any general function $g(x)$.

$$||g||^2 = \int_a^b w(x) g(x)^2 dx.$$

3. Suppose s is a root of the equation $f(x) = 0$ with multiplicity 2 (double root).

   (a) Show that Newton's method converges to this root linearly.

   (b) Modify Newton's method such that the sequence $\{x_n\}$ obtained from Newton's iterations converges to s quadratically.

   (c) By using Newton's method find $\sqrt[5]{32}$. Take starting value $x_0 = 1.8$ and carry out at most 3 iterations.

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4. You are required to obtain numerical integration formulas for

$$\int_{-1}^{1} f(x)dx$$

(a) Using only $f(1), f'(-1)$ and $f''(0)$ find an approximation to $\int_{-1}^{1} f(x)dx$ which is exact for all quadratic polynomials. i.e. $\int_{-1}^{1} f(x)dx = Af(1) + Bf'(-1) + Cf''(0)$.

(b) Derive a 3- point Gaussian quadrature formula

$$\int_{-1}^{1} f(x)dx = A_0 f(x_0) + A_1 f(x_1) + A_3 f(x_3).$$

(c) Show that the formula obtained in part (b) is exact for all polynomials of degree 5.
