September, 2015

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. Let $x_0, x_1, \cdots x_n$ be distinct real numbers and $l_k(x)$ be the Lagrange's basis polynomials. **Show that:**

   (a) For any polynomial $p(x)$ of degree $(n+1)$,

   $$p(x) - \sum_{k=0}^{n} p(x_k) l_k(x) = \frac{1}{(n+1)!} p^{(n+1)}(x) \phi_n(x)$$

   where $\phi_n(x) = \prod_{k=0}^{n} (x - x_k)$.

   (b) If $x_0, \cdots, x_n$ are the roots of the Gauss-Legendre polynomial of degree $(n+1)$ in the interval $[-1, 1]$, then

   $$\int_{-1}^{1} l_i(x) l_j(x) dx = 0 \qquad \text{for} \quad i \neq j\,.$$

2. The function f has a continuous fourth derivative on $[-1, 1]$. Construct the **Hermite** interpolation polynomial of degree 3 for f using the interpolation points $x_0 = -1$ and $x_1 = 1$. Deduce that

   $$\int_{-1}^{1} f(x)\, dx = [f(-1) + f(1)] + \frac{1}{3}[f'(-1) - f'(1)] + E$$

   where

   $$|E| \leq \frac{2}{45} \max_{x \in [-1,1]} |f^{(4)}(x)|$$

3. Evaluate the following integral

   $$\int_{1}^{\infty} e^{-x} x^2 dx$$

   using proper <u>Gaussian quadrature</u>.

   Hint: You may take

   $$\sum_{i=1}^{n} A_i x_i^2 = 2 \quad , \quad \sum_{i=1}^{n} A_i x_i = 1 \quad , \quad \sum_{i=1}^{n} A_i = 1$$

   in your Gaussian quadrature where x_i and A_i are the points and weights of the integration, respectively.

4. Consider the fixed point iteration method

$$x_{n+1} = g(x_n) \tag{1}$$

(a) State the necessary conditions for existence and uniqueness of a fixed point $x = \alpha$ in (1), and deduce the criteria that determines the order of convergence.

(b) Consider instead the fixed-point iteration

$$x_{n+1} = G(x_n) = x_n - \frac{(g(x_n) - x_n)^2}{g(g(x_n)) - 2g(x_n) + x_n} \tag{2}$$

Show that if α is a fixed point of $g(x)$, then it also a fixed point of $G(x)$.

(c) Consider the function $g(x) = x^2$, and deduce the convergence properties for both fixed point methods around the roots $x = 0$ and $x = 1$.