September, 2016

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. (a) Show that the function $f : (0,1) \to (0,\infty)$ defined by $f(x) = -\ln x$ has a unique fixed point $s \in (0,1)$.
   (b) Show, however, that fixed point iteration on $f(x)$ does NOT converge to s.
   (c) Reformulate the problem so that s is the unique fixed point of another function g for which the fixed point iteration converged to s for any $x_0 \in (0,1)$.

2. (a) Write down the conditions that should be satisfied so that the following function is a **natural cubic spline** on the interval $[0,2]$:

$$S(x) = \begin{cases} f_1(x) & : x \in [0,1], \\ f_2(x) & : x \in [1,2] \end{cases}$$

   (b) Determine the values of the coefficients a, b, c, d so that the following

$$S(x) = \begin{cases} x^2 + x^3, & : x \in [0,1], \\ a + bx + cx^2 + dx^3 & : x \in [1,2] \end{cases}$$

   is a **cubic spline** which has the property $S_1'''(x) = 12$.

3. Let $\langle f, g\rangle = \displaystyle\int_a^b w(x)f(x)g(x)dx$, where $w(x) \geq 0$ is a given weight function on $[a,b]$.
   (a) Prove that the sequence of polynomials defined below is orthogonal with respect to the inner product $\langle .,. \rangle$:

$$p_n(x) = (x - a_n)p_{n-1}(x) - b_n p_{n-2}(x), \quad n > 1,$$

   with

$$p_0(x) = 1,\ p_i(x) = x - a_i,$$
$$a_n = \langle xp_{n-1}, p_{n-1}\rangle / \langle p_{n-1}, p_{n-1}\rangle$$
$$b_n = \langle xp_{n-1}, p_{n-2}\rangle / \langle p_{n-2}, p_{n-2}\rangle.$$

   (b) Let $w(x) = 1 - x$ and $a = 0,\ b = 1$. Find the Gaussian quadrature for the integral $\displaystyle\int_0^1 (1-x)f(x)g(x)dx$, which has algebraic degree of accuracy there. Use the general theory by constructing the corresponding orthogonal polynomials.

4. Let $f \in C^6[-1,1]$.

(a) Construct the Hermite interpolating polynomial $p(x)$ on the interval $[-1,1]$ such that

$$p(x_i) = f(x_i),\ p'(x_i) = f'(x_i) \text{ for } x_i = -1, 0, 1$$

(b) Give a formula for the interpolation error

$$E(f) = p(x) - f(x).$$

(c) Show that the quadrature formula

$$\int_{-1}^{1} f(t)dt \approx \frac{7}{15}f(-1) + \frac{16}{15}f(0) + \frac{7}{15}f(1) + \frac{1}{15}f'(-1) - \frac{1}{15}f'(1)$$

is exact for all polynomials of degree ≤ 5.