September, 2018

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. Newton's method iteration applied to the equation $f(x) = x^3 - x = 0$ takes the form

$$x_{n+1} = x_n - \frac{x_n^3 - x_n}{3x_n^2 - 1}, \quad , n = 0, 1, 2, \ldots$$

   (a) Study the behavior of the iteration when $x_0 > 1/\sqrt{3}$ to conclude that the sequence $\{x_0, x_1, \ldots\}$ approaches the same root as long as you choose $x_0 > 1/\sqrt{3}$

   (b) Assume $-\alpha < x_0 < \alpha$. For what number α does the sequence always approach 0?

   (c) For an arbitrary $f(x)$, suppose that $f'(x)f''(x) \neq 0$ in an interval $[a, b]$, where $f''(x)$ is continuous and $f(a)f(b) < 0$. Show that if $f(x_0)f''(x_0) > 0$, for $x_0 \in [a, b]$, then the sequence $\{x_0, x_1, \ldots\}$ genererated by Newton's method converges monotonically to a root $\alpha \in [a, b]$.

2. (a) Explain how to find weights w_i and nodes x_i such that the quadrature $\displaystyle\sum_{i=0}^{n} w_i f(x_i)$ give the exact solution to $\displaystyle\int_0^\infty e^{-x} f(x)dx$ whenever f is a polynomial of degree $\leq n$.

   (b) Find the second-order Laguerre polynomial using Gram-Schmidt.

   (c) By using part (b), find nodes x_0 and x_1 and weights w_0 and w_1 such that the quadrature $\sum_{i=0}^{n} w_i f(x_i)$ gives exact solution to $\displaystyle\int_0^\infty e^{-x} f(x)dx$ whenever f is a polynomial of degree ≤ 1.

   (d) let $I(f) = \displaystyle\int_a^b f(x)dx$ and $I_n(f) = \displaystyle\sum_{i=0}^{n} w_i f(x_i)$. Prove that if I_n integrates to degree n exactly and w_i are all positive then the quadrature is convergent for any $f \in C([a, b])$, i.e. $\displaystyle\lim_{n\to\infty} I_n(f) = I(f)$.

3. There are three unrelated parts in this question.

   (a) Write a quadratic spline interpolant for the data $(x, y) = \{(-1, 2), (0, 1), (0.5, 0), (1, 1), (2, 2), (2.5, 3)\}$

   (b) Find a first order method for approximating $f''(x)$ that uses the data $f(x - h), f(x), f(x + 3h)$. Find the error term.

   (c) If we interpolate the function $f(x) = e^{x-1}$ with a polynomial $p(x)$ of degree 12 using **thirteen** nodes $x_j \in [-1, 1]$, find an upper bound for $|f(x) - p(x)|$ on $[-1, 1]$?

4. (a) Write the Hermite interpolation polynomial to $f(x)$ based on the values of $f(a), f'(a), f(b)$.

   (b) Based on the result of part (a), write an approximation of

$$\int_a^b f(x)\, dx$$