September, 2020

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. Consider the error $e(x) = f(x) - p_n(x)$ in Chebyshev interpolation $p_n(x)$ for a function $f(x)$ which is continuous in a general interval $[a, b]$. The roots of $k - th$ degree Chebyshev polynomial $T_k(x)$ are given by

$$x_i = \cos\frac{(2i+1)\pi}{2k}, \quad i = 0, 1, \cdots k-1 \quad \text{when} \quad x \in [-1, 1].$$

Show that

$$\| e(x) \|_\infty \leq \frac{2}{(n+1)!}\Big(\frac{b-a}{4}\Big)^{n+1} \| f^{n+1} \|_\infty \qquad \text{when} \qquad x \in [a, b].$$

2. Let the function $f(x)$ be defined as

$$f(x) = \begin{cases} c_1 x + c_2(x-1)^3, & x \in (-\infty, 0], \\ c_3 x + c_4(x-1)^3, & x \in [0, 1], \\ c_5 x + c_6(x-1)^3, & x \in (1, \infty). \end{cases}$$

   (a) Determine the relation between the coefficients $c_1, \cdots, c_6$ for which $f(x)$ is a cubic spline.

   (b) Find the values of $c_1, \cdots, c_6$ so that the cubic spline from part (a) interpolates

x	0	1	2
$f(x)$	-2	2	7

3. Let $w(x)$ be a positive continuous weight function on $[a, b]$ with

$$\int_a^b w(x)dx = 1, \quad \int_a^b xw(x)dx = 2, \quad \int_a^b x^2 w(x)dx = 5,$$

$$\int_a^b x^3 w(x)dx = 10, \quad \int_a^b x^4 w(x)dx = 30\,.$$

   (a) Derive the two-point Gaussian quadrature formula

$$\int_a^b f(x)w(x)dx \approx A_1 f(x_1) + A_2 f(x_2)$$

   which is exact for all polynomials of degree≤ 3.

   (b) Find the error for the Gaussian quadrature formula obtained in part (a).

4. Consider the Newton method to find a root α of the equation $f(x) = 0$;

$$x_{n+1} = x_n - \frac{f(x_n)h}{f(x_n+h) - f(x_n)} \quad \text{where} \quad 0 < h < 1/5.$$

Assume that f has two continuous derivatives. Also, for all x,

$$1 \leq f'(x) \leq 2 \quad \text{and} \quad |f''(x)| \leq 2.$$

If the first guess is quite good, $|x_0 - \alpha| \leq 1/5$. Show that $x_n \to \alpha$. (You may find it useful to show that first that $|x_n - \alpha| \leq 1/5$, for all n).