1. Consider $f(x) = (x - a)^n$ for some positive integer $n$ and some real number $a$

   (a) Find the sequence $\{x_i\}$ generated by the Newton and show that

   $$x_{i+1} - a = \left(1 - \frac{1}{n}\right)(x_i - a)$$

   (b) Find the order of convergence of the sequence $\{x_i\}$.

   (c) Is this order compatible with the order of Newton’s method? Give an explanation.
2. Calculate a third order interpolating polynomial through the points \((0, 0), (1, -2), (2, 0)\) and \((3, 12)\) using Newton’s Forward Divided Difference method. Give the table of differences, and compute the error of approximation of the resulting polynomial for \(x = 4\). Would you get a different result using Newton’s backward divided differences?
3. Let \( < h, g > = \int_{a}^{b} \omega(x)h(x)g(x)dx \) for \( h(x) \) and \( g(x) \) in \( C[a,b] \) and \( \omega(x) \) is a continuous positive weight function on \((a,b)\). Let \( ||h|| = < h, h >^{1/2} \).

(a) If \( f(x) \in C[a,b] \), then the polynomial \( p_n^*(x) \in P_n \) which satisfies \( ||f - p_n^*|| \leq ||f - p|| \) \( \forall p(x) \in P_n \) is given by

\[
p_n^*(x) = \sum_{j=0}^{n} < f, p_j > p_j(x)
\]

where \( \{p_j(x)\}_{j=0}^{n} \) is the orthonormal set of polynomials generated by the Gram-Schmidt process with respect to the inner product given above (\( P_n \) is the set of \( n \)-th degree polynomials).

(b) Show that the remainder function \( (f(x) - p_n^*(x)) \) is orthogonal to every polynomial in \( P_n \).

(c) Show that

\[
||f - p_n^*||^2 = ||f||^2 - \sum_{j=0}^{n} < f, p_j >^2.
\]
4. Find an approximate formula for the evaluation of the integral
\[ \int_0^1 f(x)x^{-1/2}dx \]
that is exact for all polynomial of degree one of the form
\[ I(f) = c_1 f(0) + c_2 f(1). \]
Determine the Peano kernel and the error term.