5.2.2007

Graduate Preliminary Examination
Numerical Analysis II

Duration: 3 hours

1. Consider $f(x) = (x-a)^n$ for some positive integer n and some real number a

   (a) Find the sequence $\{x_i\}$ generated by the Newton and show that

   $$x_{i+1} - a = \left(1 - \frac{1}{n}\right)(x_i - a)$$

   (b) Find the order of convergence of the sequence $\{x_i\}$.

   (c) Is this order compatible with the order of Newton's method? Give an explanation.

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2. Calculate a third order interpolating polynomial through the points $(0, 0), (1, -2)$ $(2, 0)$ and $(3, 12)$ using Newton's Forward Divided Difference method. Give the table of differences, and compute the error of approximation of the resulting polynomial for $x = 4$. Would you get a different result using Newton's backward divided differences?

3. Let $< h, g >= \int_a^b \omega(x)h(x)g(x)dx$ for $h(x)$ and $g(x)$ in $C[a,b]$ and $\omega(x)$ is a continuous positive weight function on (a,b). Let $||h|| =< h, h >^{1/2}$.

   (a) If $f(x) \in C[a,b]$, then the polynomial $p_n^*(x) \in P_n$ which satisfies $||f - p_n^*|| \leq ||f - p|| \quad \forall p(x) \in P_n$ is given by

   $$p_n^*(x) = \sum_{j=0}^{n} < f, p_j > p_j(x)$$

   where $\{p_j(x)_{j=0}^n$ is the orthonormal set of polynomials generated by the Gram-Schmidt process with respect to the inner product given above (P_n is the set of $n - th$ degree polynomials).

   (b) Show that the remainder function $(f(x) - p_n^*(x))$ is orthogonal to every polynomial in P_n.

   (c) Show that

   $$||f - p_n^*||^2 = ||f||^2 - \sum_{j=0}^{n} < f, p_j >^2 .$$

4. Find an approximate formula for the evaluation of the integral

$$\int_0^1 f(x)x^{-1/2}dx$$

that is exact for all polynomial of degree one of the form

$$I(f) = c_1 f(0) + c_2 f(1).$$

Determine the Peano kernel and the error term.