Graduate Preliminary Examination  
Numerical Analysis II  
Duration: 3 Hours  

1. The following integration formula is Gaussian quadrature type  
\[ \int_{-1}^{1} f(x) \, dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \]

(a) Derive this formula.  
(b) Determine a formula for the integration  
\[ \int_{a}^{b} f(t) \, dt \]

(c) By using part (a) and (b), evaluate  
\[ \int_{0}^{\pi/2} t \, dt \]

2. Assume that \( f \) be a 3 times continuously differentiable function near a root \( \alpha \). Show that the iterative process  
\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x_n)}{2[f'(x_n)]^3} \]

is a third order process for solving \( f(x) = 0 \).

3. Estimate the multiple integral  
\[ I = \int_{0}^{1} \int_{-1}^{1} e^{x} \left(x + \frac{1}{y}\right) \, dy \, dx \]

numerically by using  
(a) Trapezoidal rule in both \( x \) and \( y \) directions.  
(b) Composite Trapezoidal rule in \( x \) direction and Trapezoidal rule in \( y \) direction.
4. By using Newton form of an interpolating polynomial show that

(a) If \( p(x) \in \mathcal{P}_n \) (the set of all n-th degree polynomials) interpolates a function \( f \) at a set of \( n + 1 \) distinct nodes \( x_0, x_1, \ldots, x_n \) and if \( t \) is a point different from the nodes, then

\[
 f(t) - p(t) = f[x_0, x_1, \ldots, x_n, t] \prod_{j=0}^{n} (t - x_j). 
\]

(b) If \( f \in C^n[a, b] \) and if \( x_0, x_1, \ldots, x_n \) are distinct points in \( [a, b] \) then there exists a point \( \eta \in (a, b) \) such that

\[
 f[x_0, x_1, \ldots, x_n] = \frac{f^{(n)}(\eta)}{n!}. 
\]

(c) If \( f \) is a polynomial of degree \( k \), then for \( n > k \),

\[
 f[x_0, x_1, \ldots, x_n] = 0. 
\]