February, 2015

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. For finding the square root of 3, the nonlinear equation $f(x) = x^2 - 3 = 0$ is given. For each of the functions below determine whether the corresponding fixed-point iteration scheme

$$x_{k+1} = g_i(x_k), \quad i = 1, 2, 3, \quad k = 0, 1, 2, \cdots$$

is locally convergent to $\sqrt{3}$. Explain your reasoning in each case.

(a) $g_1(x) = 3 + x - x^2$

(b) $g_2(x) = 1 + x - \dfrac{x^2}{3}$

(c) $g_3(x) = x + x^2 - 3$.

Carry out 2 iterations with the convergent $g_i(x)$ to find $\sqrt{3}$ approximately correct to <u>two</u> decimal places. What is the order of convergence? Why?

2. Consider the numerical quadrature rule to approximate $\int_0^1 f(x)\,dx$ given by

$$\int_0^1 f(x)\,dx \approx af(0) + bf(x_1).$$

(a) Find the maximum possible degree of precision you can attain by appropriate choices of a, b and x_1.

(b) With such choices of a and b, approximate $\displaystyle\int_0^1 x^3\,dx$ and compare with the exact value.

3. Suppose $H(x)$ is a piecewise cubic polynomial interpolating a function $f(x)$ as follows:

$$H(\xi_i) = f(\xi_i), \quad H'(\xi_i) = f'(\xi_i), \quad i = 0, 1, \ldots, , N,$$

where ξ_i's form a partition of $[a, b]$ such that

$$\begin{aligned} a &= \xi_0 < \xi_1 < \cdots < \xi_N = b \\ h &= \xi_i - \xi_{i-1}, \quad i = 1, 2, \ldots, N \end{aligned}$$

Define $R(f; x)$ to be the error function given by

$$R(f; x) = f(x) - H(x)$$

and assume that $f(x)$ is in $C^4([a, b])$.

(a) Show that

$$\frac{d^4}{dx^4} R(f; x) = \frac{d^4}{dx^4} f(x)$$

(b) Show that for $x \in [\xi_i, \xi_{i+1}]$, there exists a $y \in (\xi_i, \xi_{i+1})$ such that

$$R(f; x) = \frac{(x - x_i)^2 (x - x_{i+1})^2}{4!} f^4(y)$$

(c) Show that

$$\max_{a \le x \le b} |R(f, x)| \le Ch^4$$

4. Let

$$I_n(f) = \sum_{k=1}^{n} w_{n,k} f(x_n, k), \quad a \le x_{n,k} \le b \tag{1}$$

be a sequence of integration rules.

(a) Suppose

$$\lim_{n\to\infty} I_n(x^k) = \int_a^b x^k dx, \quad k = 0, 1, \ldots \tag{2}$$

and

$$\sum_{k=1}^{n} |w_{n,k}| \le M \quad n = 1, 2, \ldots \tag{3}$$

for some constant M. Show that

$$\lim_{n\to\infty} I_n(f) = \int_a^b f(x)dx$$

for all $f \in C[a, b]$. (Hint: use Weierstrass approximation theorem)

(b) Show that if all $w_{n,k} > 0$ then (2) implies (3).