1. For finding the square root of 3, the nonlinear equation \( f(x) = x^2 - 3 = 0 \) is given. For each of the functions below determine whether the corresponding fixed-point iteration scheme

\[
x_{k+1} = g_i(x_k), \quad i = 1, 2, 3, \quad k = 0, 1, 2, \ldots
\]

is locally convergent to \( \sqrt{3} \). Explain your reasoning in each case.

(a) \( g_1(x) = 3 + x - x^2 \)
(b) \( g_2(x) = 1 + x - \frac{x^2}{3} \)
(c) \( g_3(x) = x + x^2 - 3 \).

Carry out 2 iterations with the convergent \( g_i(x) \) to find \( \sqrt{3} \) approximately correct to two decimal places. What is the order of convergence? Why?

2. Consider the numerical quadrature rule to approximate \( \int_0^1 f(x) \, dx \) given by

\[
\int_0^1 f(x) \, dx = a f(0) + b f(x_1).
\]

(a) Find the maximum possible degree of precision you can attain by appropriate choices of \( a, b \) and \( x_1 \).

(b) With such choices of \( a \) and \( b \), approximate \( \int_0^1 x^3 \, dx \) and compare with the exact value.

3. Suppose \( H(x) \) is a piecewise cubic polynomial interpolating a function \( f(x) \) as follows:

\[
H(\xi_i) = f(\xi_i), \quad H'(\xi_i) = f'(\xi_i), \quad i = 0, 1, \ldots, N,
\]

where \( \xi_i \)'s form a partition of \([a, b]\) such that

\[
a = \xi_0 < \xi_1 < \cdots < \xi_N = b
\]
\[
h = \xi_i - \xi_{i-1}, \quad i = 1, 2, \ldots, N
\]

Define \( R(f; x) \) to be the error function given by

\[
R(f; x) = f(x) - H(x)
\]

and assume that \( f(x) \) is in \( C^4([a, b]) \).

(a) Show that

\[
\frac{d^4}{dx^4} R(f; x) = \frac{d^4}{dx^4} f(x)
\]

(b) Show that for \( x \in [\xi_i, \xi_{i+1}] \), there exists a \( y \in (\xi_i, \xi_{i+1}) \) such that

\[
R(f; x) = \frac{(x - \xi_i)^2 (x - \xi_{i+1})^2}{4!} f^{(4)}(y)
\]

(c) Show that

\[
\max_{a \leq x \leq b} |R(f, x)| \leq C h^4
\]
4. Let

\[ I_n(f) = \sum_{k=1}^{n} w_{n,k}f(x_n,k), \quad a \leq x_n \leq b \]  

(1)

be a sequence of integration rules.

(a) Suppose

\[ \lim_{n \to \infty} I_n(x^k) = \int_{a}^{b} x^k \, dx, \quad k = 0, 1, \ldots \]  

(2)

and

\[ \sum_{k=1}^{n} \left| w_{n,k} \right| \leq M \quad n = 1, 2, \ldots \]  

(3)

for some constant \( M \). Show that

\[ \lim_{n \to \infty} I_n(f) = \int_{a}^{b} f(x) \, dx \]

for all \( f \in C[a,b] \). (Hint: use Weierstrass approximation theorem)

(b) Show that if all \( w_{n,k} > 0 \) then (2) implies (3).