1. Given the functional
\[ L(f) = f(1) - 2f(0) + f(-1) \]
for all functions \( f \in C[-1, 1] \).

(a) Show that \( L(p_1(x)) = 0 \) for all first-degree polynomials \( p_1(x) \) on \([-1, 1]\).

(b) Show that
\[ |L(f)| \leq \|L\| \cdot E_1(f) \]
for all \( f \in C[-1, 1] \) where
\[ \|L\| = \sup_{\|f\|=1} |L(f)|, \quad E_1(f) = \inf \|f - p_1\|_{\infty}, \]
and
\[ \|f - p_1\|_{\infty} = \max_{x \in [-1, 1]} |f(x) - p_1(x)|. \]

(c) Show that the curve \( f(x) = e^x \) can not be approximated by a straight line in \([-1, 1]\) with an error which is less than \( \frac{e - 2 + e^{-1}}{4} \approx 0.271 \).

2. Let \( f \) be a twice continuously differentiable function on \( \mathbb{R} \) which has a unique root on the interval \([0, 9]\). Suppose that for all \( x \in [0, 9], \ f'(x) \geq 2 \) and \( |f''(x)| \leq 6 \) are satisfied.

(a) Determine if the secant iteration converge for any \( x_0, x_1 \in [0, 9] \).

(b) If the secant iteration converge, find the number of iterations required to get an error that is less than \( 10^{-8} \).

(c) If the convergence of the secant method is not guaranteed for all \( x \in [0, 9] \), decide the number of steps of bisection needed before convergence will be guaranteed for the secant iteration. Explain your answer.

3. (a) Let \( \phi(h) \) be an approximation for \( K \) for each \( h > 0 \) such that
\[ K = \phi(h) + C_1h + C_2h^2 + C_3h^3 + \ldots \]
with arbitrary constants \( C_1, C_2 \) and \( C_3 \). Use the values \( \phi(h), \phi(h/3), \phi(h/9) \) to produce an approximation to \( K \) of order \( h^3 \) (i.e. \( O(h^3) \)).

(b) Consider the difference formula
\[ f'(x_0) = \frac{1}{h} (f(x_0 + h) - f(x_0)) - \frac{h}{2} f''(x_0) - \frac{h^2}{6} f'''(x_0) + O(h^3) \]
for \( f'(x) \) at \( x_0 \). Use this formula with the extrapolation of part (a) to derive an \( O(h^3) \) formula for \( f'(x_0) \).
4. Consider numerical integration

\[ I_{\text{app}} = \sum_{j=0}^{n} \beta_j f(x_j), \quad x_j \in [-1, 1] \]

for the integral

\[ I = \int_{-1}^{1} f(x)w(x)dx \]

where \( w \) is positive weight function in [-1,1]. Let

\[ \Omega_{n+1}(x) = \prod_{j=1}^{n} (x - x_j) \]

denote the polynomial of degree \( n + 1 \) associated with the distinct quadrature nodes \( x_0, x_1, x_2, \ldots, x_n \). Prove that

\[ \int_{-1}^{1} \Omega_{n+1}(x)p(x)w(x)dx = 0 \]

for any polynomial of degree less or equal to \( m - 1 \) if and only if the quadrature formula is exact for all polynomials of degree less or equal \( n + m \).