PRELIMINARY EXAM PROBLEMS
Differential Equations (ODE), 3 hours, 12.09.2006

1. Consider the differential equation
\[ y'' + q(x)y = 0, \quad (1) \]
where \( q : [\alpha, \beta] \to \mathbb{R} \) is a continuous function such that \( 0 < m \leq q(x) \leq M \). Let \( \{x_1, x_2, \ldots, x_n\} \) be the zeros of a solution \( y(x) \) such that \( \alpha \leq x_1 < x_2 < \ldots < x_n \leq \beta \).

Show that:
(a) \( \frac{\pi}{\sqrt{M}} \leq x_{i+1} - x_i \leq \frac{\pi}{\sqrt{m}}, i = 1, 2, \ldots, n - 1; \)
(b) \( \frac{\sqrt{m}}{\pi}(\beta - \alpha) < n + 1. \)

2. Applying the differentiable dependence of solutions on the initial value estimate the deviation of a solution \( y(t) = y(x, 0, y_0) \) of the equation \( y' = y + \sin y \) on \([0, 1]\) if the initial value is changed from 0 to \( y_0 \) and \( |y_0| < 0.01. \)

3. (a) Find all values of a parameter \( a \in \mathbb{R} \) such that the system
\[ x' = 2y - 4x + 1, \quad y' = 2x - y + a \]
has solutions bounded on \( \mathbb{R} \).
(b) Define all these bounded solutions.
(c) Are these solutions stable?

4. For the initial value problem
\[ y' = \lambda + \cos y, y(0) = 0, \]
find an upper estimate for \( |y(x, \lambda_1) - y(x, \lambda_2)| \) and deduce that \( y(x, \lambda) \) is continuous.