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TMS

Fall 2009

Ordinary Differential Equations

1. Consider the differential equation

$$y' = yg(t, y),$$

where g and $\frac{\partial g}{\partial y}$ are defined and continuous for all $(t, y) \in R^2$. Show that

(a) if $y = y(t)$, $t \in (a, b)$, is a solution satisfying $y(t_0) = y_0 > 0$, $t_0 \in (a, b)$, then $y(t) > 0$ for all $t \in (a, b)$

(b) if $y = y(t)$, $t \in (a, b)$, is a solution satisfying $y(t_0) = y_1 < 0$, $t_0 \in (a, b)$, then $y(t) < 0$ for all $t \in (a, b)$

2. Consider the linear system

$$\dot{x} = A(t)x,$$

where A is a continuous for all $t \in J \subseteq R$.

Let $\Phi(t, t_0)$ be a matrix solution of the above system satisfying $\Phi(t_0, t_0) = I$.

(a) Show that $x(t) = \Phi(t, t_0)x_0$ is the unique solution of the system satisfying $x(t_0) = x_0$.

(b) Show that $\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)$, $t_1 \in J$.

(c) Show that $\frac{\partial \Phi(t,s)}{\partial s} = -\Phi(t, s)A(s)$, $t, s \in J$.

(d) Let $X(t) := \Phi(t, 0)$, $0 \in J$. Show that $\Phi(t, t_0) = X(t - t_0)$ if and only if A is a constant matrix.

3. Find the Folquet multipliers for the periodic system

$$\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & \frac{\cos t + \sin t}{2 + \sin t - \cos t} \end{bmatrix} x$$

and deduce that there is a periodic s9olution? What is the periodic solution?

4. Show that the zero solution of

$$\dot{x} = -2x + \frac{\sin t}{t^2 + 1} x^3$$

is uniformly asymptotically stable.