M E T U
Department of Mathematics
GRADUATE PRELIMINARY EXAM
Ordinary Differential Equations – September, 2014

Last Name :
Name :

Q.1 Consider the linear ODE: $x' = a(t)x + b(t)$, where $a(t)$ and $b(t)$ are continuous real functions on $t \geq 0$. Prove the following statements:

(a) The solution, satisfying $x(t_0) = x_0 \in \mathbb{R}$ for any $t_0 \geq 0$, is given by

$$x(t) = x_0 e^{\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} b(u) e^{\int_{u}^{t} a(s)ds} du$$

(b) If $a(t) \leq -m < 0$ and $b(t)$ is bounded on $t \geq 0$, then any solution is bounded on $t \geq 0$.

(c) If $a(t) \geq m > 0$ and $b(t)$ is bounded on $t \geq 0$, then there exists one and only one solution bounded on $t \geq 0$, which is given by

$$x(t) = -\int_{t}^{\infty} b(u) e^{-\int_{t}^{u} a(s)ds} du$$

(d) If $\lim a(t) = -A$ with $A > 0$ and $\lim b(t) = B$ as $t \to \infty$, then any solution of the linear ODE satisfies $\lim x(t) = B/A$ as $t \to \infty$.

Q.2 Let $\mathbf{A}(t)$ be an $n \times n$ continuous matrix for all $t \in \mathbb{R}$. Let $\Psi(t)$ be a matrix solution of $\mathbf{X}' = \mathbf{A}(t)\mathbf{X}$ with $\Psi(0) = \mathbf{I}_n$. Show that $\Psi(t)\Psi^{-1}(s) = \Psi(t-s)$ for all $t, s \in \mathbb{R}$, if and only if $\mathbf{A}(t)$ is a constant matrix.

Q.3 Consider the nonlinear system

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= y \\ \mathrm{d}y/\mathrm{d}t &= -w^2 \sin x - \gamma y \end{aligned}$$

where γ and w are real constants.

(a) Find the critical points (equilibrium solutions), and deduce that the origin is an isolated critical point of the system.

(b) Using the **linear approximation**, examine the stability properties of the critical point at the origin.

(c) Explain why the trajectories of the linear system are good approximations to those of the nonlinear system, at least near the origin.