M E T U
Department of Mathematics
GRADUATE PRELIMINARY EXAM
Ordinary Differential Equations – September, 2015

Last Name :

Name :

Q.1 Use *Sturm Comparison Theory* to find the least possible number of zeros of a nontrivial solution of $y'' + t^2 y = 0$ on $(0, 5\pi)$. At most, how many zeros can have such a solution on $[0, 5\pi]$?

Q.2 Use *Green's formula* to find the differential operator adjoint to

$$L = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + a_1(t)\frac{\mathrm{d}}{\mathrm{d}t} + a_0(t),$$

where a_0 and a_1 are real valued continuous functions on $t \in [a, b]$. Hence show that L is NOT formally self-adjoint. Then determine a function $\mu(t)$ appropriately to see that the operator $\mu(t)L$ is formally self-adjoint.

Q.3 Let $\mathbf{\Phi}(t)$ and $\mathbf{\Psi}(t)$ be two fundamental matrices for the linear homogeneous system $\mathbf{x}' = \mathbf{A}(t)\mathbf{x}$, where $\mathbf{A}(t)$ is an $n \times n$ continuous real matrix on $t \in (a, b)$.
(a) Show that there is an invertible constant matrix $\mathbf{C}$ such that $\mathbf{\Phi}^{-1}(t)\mathbf{\Psi}(t) = \mathbf{C}$.
(b) If $\mathbf{W}(t)$ is a fundamental matrix for the adjoint system $\mathbf{y}' = -\mathbf{A}^T(t)\mathbf{y}$, show also that $\mathbf{W}^T(t)\mathbf{\Phi}(t) = \mathbf{C}$.

Q.4 Consider the nonlinear system

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= 2y \\ \mathrm{d}y/\mathrm{d}t &= -4\cos(x + \pi/2) + y \end{aligned}$$

(a) Find the critical points (equilibrium solutions), and deduce that the origin is an isolated critical point of the system.
(b) Using the **linear approximation**, examine the stability properties of the critical point at the origin.
(c) Explain why the trajectories of the linear system are good approximations to those of the nonlinear system, at least near the origin.