Q.1 Use Sturm Comparison Theory to find the least possible number of zeros of a nontrivial solution of \( y'' + t^2 y = 0 \) on \((0, 5\pi)\). At most, how many zeros can have such a solution on \([0, 5\pi]\)?

Q.2 Use Green's formula to find the differential operator adjoint to

\[
L = \frac{d^2}{dt^2} + a_1(t) \frac{d}{dt} + a_0(t),
\]

where \(a_0\) and \(a_1\) are real valued continuous functions on \(t \in [a, b]\). Hence show that \(L\) is NOT formally self-adjoint. Then determine a function \(\mu(t)\) appropriately to see that the operator \(\mu(t)L\) is formally self-adjoint.

Q.3 Let \(\Phi(t)\) and \(\Psi(t)\) be two fundamental matrices for the linear homogeneous system \(x' = A(t)x\), where \(A(t)\) is an \(n \times n\) continuous real matrix on \(t \in (a, b)\).

(a) Show that there is an invertible constant matrix \(C\) such that \(\Phi^{-1}(t)\Psi(t) = C\).

(b) If \(W(t)\) is a fundamental matrix for the adjoint system \(y' = -A^T(t)y\), show also that \(W^T(t)\Phi(t) = C\).

Q.4 Consider the nonlinear system

\[
\begin{align*}
\frac{dx}{dt} &= 2y \\
\frac{dy}{dt} &= -4\cos(x + \pi/2) + y
\end{align*}
\]

(a) Find the critical points (equilibrium solutions), and deduce that the origin is an isolated critical point of the system.

(b) Using the linear approximation, examine the stability properties of the critical point at the origin.

(c) Explain why the trajectories of the linear system are good approximations to those of the nonlinear system, at least near the origin.