PRELIMINARY EXAM PROBLEMS
Differential Equations (ODE), 2004/2

1. Consider the following IVP
$$y' = y\cos(x^2 + y^2), \quad y(0) = 1.$$
where $y, x \in R^1$.
a) Applying the "existence-uniquenes" theorem, determine a *specific* interval on which a unique solution is sure to exist.
b) Determine the *largest* possible interval (α, β) on which the solution is defined.
c) Explain why the solution of the IVP is always positive.
d) Is the solution strictly increasing over its interval of definition? Why? Why not?

2. a) Let $y(t)$ be a solution of $y'' - e^{-t}y = 0$.
Show that $y(t)$ can not vanish twice.

   b) Prove that every solution of $y'' + (1 + a(t))y = 0$ has infinitely many zeros, if
$$\lim_{t\to\infty} a(t) = 0.$$

3. Suppose that all solutions of $y'' + a(t)y = 0$ are bounded. Show that if $\int_0^\infty |b(t)|dt < \infty$, then all solutions of $y'' + (a(t) + b(t))y = 0$ are also bounded.

4. Using Lyapunov function show that the zero solution of the system
$$\begin{aligned} x_1' &= -2x_1x_2^2 - x_1^3, \\ x_2' &= -x_2 + x_1^2x_2 \end{aligned} \qquad (1)$$
is uniform asymptotically stable.