PRELIMINARY EXAM PROBLEMS
Differential Equations (ODE), 3 hours, 2005/2

1. Consider the IVP
\[ x_1' = -\frac{1}{t}x_2 + \sin t, \]
\[ x_2' = \frac{1}{t}x_1 + \cos t, \]
\[ x_1(1) = x_2(1) = 0. \]
Show that the IVP has a unique solution defined on \((0, \infty)\).

2. Let \(p(t)\) be continuous function defined on \([1, \infty)\) such that
\[ \int_1^\infty |p(t) - c|dt < \infty, \quad c > 0. \]
(a) Show that all solutions of
\[ y'' + p(t)y = 0 \]  
are bounded on \([1, \infty)\). (Hint: rewrite the equation in the form \(y'' + cy = (c - p(t))y\)).
(b) What can you say about the stability of the zero solution?
(c) Show that all solutions of (1) need not to be bounded if \(c = 0\). (Hint: \(p(t) = \frac{a}{t^2}\)).

3. Let \(A(t)\) be a continuous matrix for all \(t \in \mathbb{R}\). Let \(P(t)\) be the matrix solution of
\[ X' = A(t)X. \]
Show that \(P(t)P^{-1}(s) = P(t - s)\) for all \(t, s \in \mathbb{R}\), if and only if \(A(t)\) is a constant matrix.

4. Consider the following scalar equation
\[ x' = c(t)x, \]  
where function \(c(t) : \mathbb{R} \to \mathbb{R}\) is defined in the following way:
\[ c(t) = \begin{cases} t, & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 1 - t, & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases} \]
and \(c(t)\) is \(1\)-periodic.
(a) Prove that (2) does not have a nontrivial 1-periodic solution.
(b) Does the equation have a nontrivial solution with another period?