PRELIMINARY EXAM PROBLEMS
Differential Equations (ODE), 3 hours, 2008/2

1. Consider the system
$$\begin{aligned} x' &= -3x - 2y + \sin(t), \\ y' &= 2x - 3y + \cos(t). \end{aligned} \tag{1}$$
   (a) Evaluate the transition matrix $X(t,s)$ of the associated homogeneous system. Show that $\limsup_{t\to\infty} \|X(t,s)\| = \cancel{\infty}$; 0
   (b) Find the general solution $x(t, t_0, x_0)$ of the system;
   (c) Show that all solutions are bounded on $[0,\infty)$ functions;
   (d) Show that there exists a unique solution bounded on R;
   (e) Prove that the bounded solution is $2\pi-$periodic function.
   (f) Prove that each solution of the system is uniformly asymptotically stable.

2. ~~Evaluate~~ Estimate $|x(t,0,x_0,x_0^1)|, |x'(t,0,x_0,x_0^1)|$, for $t \in [0,T], T < \infty$, if $x(t) = x(t,0,x_0,x_0^1), x(0) = x_0, x'(0) = x_0^1$, is a solution of equation $x'' + \sin x = 0$. Consider $x_0 = 0.01, x_0^1 = -0.02, T = 10$.

   Hint: Use differentiability of solutions in initial value.

3. Assume that $u(t) \geq 0, v(t) > 0$, are continuous on $[t_0 - T, t_0], t_0 \in R, T > 0$, functions. Prove that the inequality
$$u(t) \leq c + \int_t^{t_0} u(s)v(s)ds, t \leq t_0$$
   implies
$$u(t) \leq ce^{\int_t^{t_0} v(s)ds},$$
   where $c \geq 0$ is constant.

4. Consider the following Abel's equation
$$y' = \sin(t) - y^3. \tag{2}$$
   where $t, y \in R$. Prove that as t increasing, each solution of (2) is attracted into the strip $|y| < 1 + \epsilon$, where ϵ is a fixed positive number, in a uniformly bounded time interval.