M E T U
Department of Mathematics
GRADUATE PRELIMINARY EXAM
Ordinary Differential Equations – February, 2014

Last Name :

Name :

Q.1 Consider the linear ODE: $x' = a(t)x + b(t)$, where $a(t)$ and $b(t)$ are continuous real functions on $t \geq 0$. Prove the following statements:

(a) The solution, satisfying $x(t_0) = x_0 \in \mathbb{R}$ for any $t_0 \geq 0$, is given by

$$x(t) = x_0 e^{\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} b(u) e^{\int_{u}^{t} a(s)ds} du$$

(b) If $a(t) \leq -m < 0$ and $b(t)$ is bounded on $t \geq 0$, then any solution is bounded on $t \geq 0$.

(c) If $a(t) \geq m > 0$ and $b(t)$ is bounded on $t \geq 0$, then there exists one and only one solution bounded on $t \geq 0$, which is given by

$$x(t) = -\int_{t}^{\infty} b(u) e^{-\int_{t}^{u} a(s)ds} du$$

Q.2 Let $h(t) \in \mathcal{C}([0,\infty], \mathbb{R}^+)$ and let $g(x) \in \mathcal{C}((0,\infty), \mathbb{R}^+)$. Suppose that

$$\lim_{B\to\infty} \int_{A}^{B} \frac{\mathrm{d}x}{g(x)} = +\infty, \qquad A > 0.$$

Then consider the IVP: $\dfrac{\mathrm{d}x}{\mathrm{d}t} = h(t)g(x),\ x(\tau) = \xi$ with $\tau \geq 0$ and $\xi > 0$.

(a) Show that all solutions can be continued to the right over the entire interval $\tau \leq t < \infty$.

(b) If $\int_0^\infty h(t)\mathrm{d}t < \infty$, show that any solution of the IVP has a finite limit as $t \to \infty$.

(c) If $\lim_{\epsilon\to 0^+} \int_\epsilon^1 \frac{\mathrm{d}x}{g(x)} = +\infty$, show that all solutions can be continued to the left until $t = 0$.

Q.3 Consider the linear system with constant coefficients

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= a_{11}x + a_{12}y \\ \mathrm{d}y/\mathrm{d}t &= a_{21}x + a_{22}y \end{aligned}$$

where the eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ are purely imaginary.

(a) Show that all solutions are closed trajectories (ellipses) surrounding the origin in the xy-plane. **Hint:** First observe that the eigenvalues of $\mathbf{A}$ are purely imaginary if and only if $\mathrm{tr}\mathbf{A} = 0$ and $\det\mathbf{A} > 0$. Then deduce that the system can be converted into a single equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$, which is exact.

(b) Show that the equilibrium solution is stable.

Q.4 If a nontrivial solution $\phi(t)$ of $y'' + (A + B\cos 2t)y = 0$ has $2n$ zeros in $(-\pi/2, \pi/2)$ and if $A, B > 0$, show that $A + B \geq (2n-1)^2$.