1. Solve the following initial value problem.

\[ u_x^2 - 3u_y^2 = u, \quad u(x, 0) = x^2. \]

2. Let \( D \) be the region \((0, 1) \times (0, 1) \subset \mathbb{R}^2 \) and let \( u(x, y) \in C^2(D) \cap C^0(\overline{D}) \) be a non-constant function. Set \( M = \max(u) \) in \( D \).

a) Show that if \( u(x, y) \) solves the equation

\[ \nabla^2 u(x, y) + a(x, y)u_x + b(x, y)u_y = F(x, y) \quad \text{in} \; D \]

and if \( F(x, y) > 0 \) in \( D \), then \( u(x, y) < M \) for all \((x, y) \in D\).

b) True or false? The same conclusion holds if \( u(x, y) \) solves the equation \( u_{xy} = 0 \) in \( D \). (Prove the statement or give a counter example).

3. Consider the following Dirichlet problem.

\[ \nabla^2 u = 0 \quad \text{in} \; \Omega = \{(r, \theta) : r > 1, 0 < \theta < \pi/2\} \]

\[ u(r, 0) = u(r, \pi/2) = 0 \quad \text{for} \; r \geq 1 \]

\[ u(1, \theta) = \sin(2\theta) \quad \text{for} \; 0 < \theta < \pi/2. \]

a) Find the bounded solution of this problem.

b) Find an unbounded solution, if there is any.

c) Write a Neumann problem in \( \Omega \) for which the function \( u(r, \theta) \) of part (a) is a solution.
4. Let \( G(x, t) \) be the heat kernel
\[
G(x, t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}.
\]

a) Show that
\[
u(x, t) = 2 \int_0^t G(x, t - t') f(t') dt'
\]
satisfies
\[
\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{in} \quad \{(x, t) : x > 0, \ t > 0\}
\]
\[
u(x, 0) = 0, \ x > 0.
\]

Hint: Do not verify the uniform convergence of the integral, but indicate when you use this property.

b) Verify that
\[
- \frac{\partial u}{\partial x}|_{x=0} = f(t), \ t > 0.
\]

Hint: In the integral, first make the change of variable given by
\[
s^2 = \frac{x^2}{4(t - t')}.
\]