M.E.T.U

Department of Mathematics

Preliminary Exam - Sep. 2011

Partial Differential Equations

Duration : 180 min.

Each question is 25 pt.

NOTATION :
∇, Δ denote the gradient and the Laplace operators, respectively.

1. **a.** For a linear second order differential equation

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0$$

define the elliptic, parabolic and hyperbolic equation at point (x, y).

Consider the equation $u_{yy} - yu_{xx} = 0$.

b. Determine where the equation is elliptic, parabolic, hyperbolic.

c) Determine the characteristics in the region $\mathcal{H} = \{(x, y) : y > 0\}$.

2. **a.** Give the definition of Dirichlet and Neumann problems for Laplace equation in domain Ω.

b. Using the energy identity

$$\int_\Omega \left(\sum u_{x_i}^2\right) dx + \int_\Omega u\Delta u dx = \int_{\partial\Omega} u\frac{du}{dn} dS.$$

prove that if $u \in C^2(\bar{\Omega})$ is a solution of a Dirichlet problem in Ω, then it is unique.

c.) Explain if there exists a solution for each of the following problems in the unit disk $\Omega = \{(r, \theta) : r < 1\} \in \mathrm{R}^2$.

- $\Delta(u) = 0$, $u|_{\partial\Omega} = \cos(\theta)$.

- $\Delta(u) = 0$, $\frac{\partial u}{\partial n}|_{\partial\Omega} = \cos(\theta)$.

(Hint : For $f, g \in C^2(\Omega)$ one has the Stokes' formula

$$\int_\Omega (\nabla f . \nabla g) dA + \int_\Omega (f \Delta g) dA = \int_{\partial\Omega} (f \frac{\partial g}{\partial n}) dl.)$$

3. For the equation

$$\Delta u(x) + u(x) = 0 \quad x \in \mathbb{R}^3$$

find the spherically symmetric solution. That is a solution of the form $u = f(r)$, where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

(Hint: in the resulting ODE for f introduce new variable $y(r) = rf(r)$).

4. For $T \geq 0$ let $Q_T = \{(x,t) : 0 < x < L, 0 < t \leq T\}$ and $B_T = \bar{Q}_T \backslash Q_T$ ($\bar{Q}_T$ denotes the closure of Q_T). Suppose that $u(x,t)$ is continuous on $\bar{Q}_T$ and satisfies

$$u_t - a(x,t)u_{xx} - b(x,t)u_x - c(x,t)u < 0 \quad \text{on} \quad Q_T,$$

where $a(x,t) \geq 0$, $c(x,t) \leq 0$ in Q_T, and

$$u(x,t) \leq 0 \quad \text{on} \quad B_T.$$

Show that $u(x,t) \leq 0$ on $\bar{Q}_T$.

(Hint: show that $u(x,t)$ cannot have positive local maximum in Q_T.)