1. a. For a linear second order differential equation  
\[ au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0 \]
define the elliptic, parabolic and hyperbolic equation at point \((x, y)\).

Consider the equation \(u_{yy} - yu_{xx} = 0\).

b. Determine where the equation is elliptic, parabolic, hyperbolic.

c) Determine the characteristics in the region \(\mathcal{H} = \{(x, y) : y > 0\}\).

2. a. Give the definition of Dirichlet and Neumann problems for Laplace equation in domain \(\Omega\).

b. Using the energy identity
\[
\int_\Omega \left( \sum u^2_{x_i} \right) dx + \int_\Omega u\Delta u dx = \int_{\partial\Omega} u\frac{du}{dn} dS.
\]
prove that if \(u \in C^2(\bar{\Omega})\) is a solution of a Dirichlet problem in \(\Omega\), then it is unique.

c.) Explain if there exists a solution for each of the following problems in the unit disk \(\Omega = \{(r, \theta) : r < 1\} \in \mathbb{R}^2\).

- \(\Delta (u) = 0, u|_{\partial\Omega} = \cos(\theta)\).
• $\Delta(u) = 0$, $\frac{\partial u}{\partial n}|_{\partial \Omega} = \cos(\theta)$.

(Hint: For $f, g \in C^2(\Omega)$ one has the Stokes’ formula

$$\int_{\Omega} (\nabla f, \nabla g) dA + \int_{\Omega} (f \Delta g) dA = \int_{\partial \Omega} (f \frac{\partial g}{\partial n}) dl.$$ )

3. For the equation

$$\Delta u(x) + u(x) = 0 \quad x \in \mathbb{R}^3$$

find the spherically symmetric solution. That is a solution of the form $u = f(r)$, where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

(Hint: in the resulting ODE for $f$ introduce new variable $y(r) = rf(r)$).

4. For $T \geq 0$ let $Q_T = \{(x, t) : 0 < x < L, 0 < t \leq T\}$ and $B_T = \bar{Q}_T \setminus Q_T$ ($\bar{Q}_T$ denotes the closure of $Q_T$). Suppose that $u(x, t)$ is continuous on $Q_T$ and satisfies

$$u_t - a(x, t)u_{xx} - b(x, t)u_x - c(x, t)u < 0 \quad \text{on} \quad Q_T,$$

where $a(x, t) \geq 0$, $c(x, t) \leq 0$ in $Q_T$, and

$$u(x, t) \leq 0 \quad \text{on} \quad B_T.$$

Show that $u(x, t) \leq 0$ on $\bar{Q}_T$.

(Hint: show that $u(x, t)$ cannot have positive local maximum in $Q_T$.)