Partial Differential Equations

Problem 1. For the equation

$$(y^2 - x)z_x + yz_y = z$$

(a) give an example of the initial curve so that the corresponding Cauchy problem has a unique solution (do not solve the problem); (Explain)
(b) give an example of the initial curve so that the corresponding Cauchy problem has no solution. (Explain)

Problem 2. In the region $D := \{(x, y) : y > 1\}$ determine the type and transform to the canonical form the following equation

$$x^2 u_{xx} - 2xyu_{xy} + y^2 u_{yy} = 0.$$

Problem 3.
(a) Show that if $u(x, t)$ satisfies

$$\begin{array}{ll} u_{tt} + u_t = u_{xx}, & 0 < x < L,\ t > 0 \\ u(0,t) = 0,\ u(L,t) = 0 & t \geq 0 \end{array}$$

then $E(t) = \int_0^L (u_t^2 + u_x^2)dx$ is a decreasing function.
(b) Use (a) to prove uniqueness of solution for the initial-boundary value problem

$$\begin{array}{ll} u_{tt} + u_t = u_{xx}, & 0 < x < L,\ t > 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) & 0 \leq x \leq L \\ u(0,t) = a(t),\ u(L,t) = b(t), & t \geq 0 \end{array}$$

Problem 4. Does the following problem has analytic solution in the neighborhood of the point $(1, 1)$

$$u_{tt} = u_{xx}u_x^2,\ u(x,1) = \sin x,\ u_t(x,1) = e^x\,?$$

(Explaine).