Partial Differential Equations

Problem 1. Consider a partial differential equation with constant coefficients

$$\hat{a}u_{xx} + 2\hat{b}u_{xy} + \hat{c}u_{yy} + \hat{d}u_x + \hat{e}u_y + \hat{f}u = g(x,y) \qquad a \neq 0.$$

Given that the equation is parabolic.
a. Write the equation in canonical form using independent variables (a,b) (DO NOT evaluate the coefficients of canonical form). How variables (x,y) and (a,b) are related?
b. Show that the canonical form can be simplified to

$$v_{aa} + \bar{d}v_b = \bar{g}(a,b)$$

by the change of variable $u = ve^{\lambda a + \mu b}$.

Problem 2. Let D be a bounded domain in $\mathbb{R}^n$. Consider a region $\Omega = \{(x,t) : x \in D, \quad 0 < t \leq T\}$ and a function $u(x,t) \in C^2(\Omega) \cup C(\bar{\Omega})$. Show that if $u_t - a\Delta u > 0$ on Ω, where $a > 0$ then u can not assume a local minimum in Ω.

Problem 3. Consider the initial value problem

$$u_y = G(x,y,u,u_x) \qquad u(x,0) = f(x),$$

where G is twice continuously differentiable and f is three times continuously differentiable. Show that the problem has a unique solution in the neighborhood of the initial curve.

Problem 4. Let u and v be two solutions of the one-dimentional wave equation

$$u_{tt} = c^2 u_{xx}.$$

Show that $\frac{d}{dt}\int_a^b u_t v_t + c^2 u_x v_x = 0$ provided that $u = 0$, $v = 0$ for $x = a$, $t > 0$ and $x = b$, $t > 0$.