Partial Differential Equations

Problem 1. Find solution of the equation

$$xyz_x + xzz_y = yz$$

passing through the curve $z = 1 + y^2$, $x = 1$ (if exists).

Problem 2. Using Green's first identity

$$\iint\limits_{\partial D} v \frac{\partial u}{\partial n}\, dS = \iiint\limits_{D} \nabla v \cdot \nabla u\, dx + \iiint\limits_{D} v \Delta u\, dx,$$

prove the uniqueness of solution for the Robin problem

$$\Delta u = 0 \quad \text{in} \quad D, \qquad \frac{\partial}{\partial n} u(x) + a(x)u(x) = h(x) \quad \text{on} \quad \partial D,$$

provided $a(x) > 0$ on ∂D.

Problem 3. Solve the Dirichlet problem for the exterior of a circle

$$\begin{array}{ll} \Delta u = 0, & x^2 + y^2 > a^2 \\ u = h, & x^2 + y^2 = a^2 \end{array}$$

and u bounded as $x^2 + y^2 \to \infty$, given that $h(a, \theta) = \sum_{n=1}^{\infty} \alpha_n \cos 2n\theta$ (in polar coordinates (r, θ)).

Problem 4. Let $\Omega = \{(x, t) : x \in (0, a),\ t \in (0, T]\}$ and $\partial_p \Omega = \{(x, t) : x = 0, t \in [0, T] \,\text{or}\, x = a, t \in [0, T] \,\text{or}\, x \in [0, a], t = 0\}$. Suppose $u \in C^2(\Omega) \cup C(\bar{\Omega})$ satisfies $u_t - u_{xx} + cu < 0$, where $c \geq 0$, in Ω. Show that $\max\limits_{\bar{\Omega}} u = \max\limits_{\partial_p \Omega} u$ given that $\max\limits_{\partial_p \Omega} u > 0$.