Partial Differential Equations

Problem 1. Find solution of the equation

\[ xyz_x + xzz_u = yz \]

passing through the curve \( z = 1 + y^2 \), \( x = 1 \) (if exists).

Problem 2. Using Green's first identity

\[
\iint_{\partial D} v \frac{\partial u}{\partial n} dS = \iiint_D \nabla v \cdot \nabla u \, dx + \iiint_D v \Delta u \, dx,
\]

prove the uniqueness of solution for the Robin problem

\[ \Delta u = 0 \quad \text{in} \quad D, \quad \frac{\partial}{\partial n} u(x) + a(x)u(x) = h(x) \quad \text{on} \quad \partial D, \]

provided \( a(x) > 0 \) on \( \partial D \).

Problem 3. Solve the Dirichlet problem for the exterior of a circle

\[ \Delta u = 0, \quad x^2 + y^2 > a^2 \]

\[ u = h, \quad x^2 + y^2 = a^2 \]

and \( u \) bounded as \( x^2 + y^2 \to \infty \), given that \( h(a, \theta) = \sum_{n=1}^{\infty} \alpha_n \cos 2n\theta \) (in polar coordinates \((r, \theta)\)).

Problem 4. Let \( \Omega = \{(x, t) : x \in (0, a), \ t \in (0, T]\} \) and
\[ \partial_p \Omega = \{(x, t) : x = 0, t \in [0, T] \text{ or } x = a, t \in [0, T] \text{ or } x \in [0, a], t = 0\}. \]
Suppose \( u \in C^2(\Omega) \cup C(\bar{\Omega}) \) satisfies \( u_t - u_{xx} + cu < 0 \), where \( c \geq 0 \), in \( \Omega \). Show that \( \max_{\Omega} u = \max_{\partial_p \Omega} u \) given that \( \max_{\partial_p \Omega} u > 0 \).