Duration: 3 hours

1. Let \((X, S, \mu)\) be a measure space, \(T\) be a metric space. Let \(f: X \times T \to \mathbb{R}\) be a function. Assume that \(f(\cdot, t)\) is measurable for each \(t \in T\) and \(f(x, \cdot)\) is continuous for each \(x \in X\). Prove that if there exists an integrable function \(g\) such that for each \(t \in T\), \(|f(x, t)| \leq g(x)\) for \(a.a.x\), then \(F: T \to \mathbb{R}, F(t) = \int f(x, t) d\mu(x)\) is continuous.

2. Let \(G\) be a set of half-open intervals in \(\mathbb{R}\). Prove that \(\bigcup_{G \in G} G\) is Lebesgue measurable.

3. a) Let \(f_n = \sin n^2 x \in L_p[0, 1], \) where \(1 \leq p < \infty\). Show that \(f_n \to 0\) weakly, but \(f_n \not\to 0\) in measure.

b) Let \(g_n = n^2 \chi_{[0, \frac{1}{n}]} \in L_p[0, 1], \) where \(1 \leq p < \infty\). Show that \(g_n \to 0\) in measure, but \(g_n \not\to 0\) weakly.

c) Let \(A_n\) be a measurable subset of \([0, 1]\) for each \(n, \chi_{A_n} \in L_1\), and \(\chi_{A_n} \to f\) weakly in \(L_1\). Show that \(f\) is not necessarily a characteristic function of some measurable set.

4. Let \(f: \mathbb{R} \to \mathbb{R}\). If \(f \in L_1(m) \cap L_2(m)\) where \(m\) denotes the Lebesgue measure, prove that

a) \(f \in L_p(m) \ \forall \ 1 \leq p \leq 2\)

b) \(\lim_{p \to 1^+} ||f||_p = ||f||_1\).