I. a) Let $A_n$ be a sequence of measurable sets with $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Prove that 
\[ \mu(\lim A_n) = 0 \]

Hint: $\lim A_n = \bigcap_{k=1}^{\infty} \bigcup_{n \geq k} A_n$

b) Let $f \in L_p(\mu)$ and $\epsilon > 0$. Show that 
\[ \mu(\{x \in X : |f(x)| \geq \epsilon\}) \leq \epsilon^{-p} \int |f|^p d\mu \]

II. a) Show that $f(x) = \frac{1}{\sqrt{x}}$ is Lebesgue integrable over $[0, 1]$.

b) Compute $\lim_n \int_0^1 \frac{\sin x}{1+n^2 \sqrt{x}} \, dx$ and justify your calculations.

III. Assume $\mu(X) < \infty$. If $f_n$ is a sequence of measurable functions on $X$ such that 
$f_n \to f$ a.e. then prove that $f_n \to f$ [meas] also holds.

State the theorem(s) you used.

IV. Assume that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are two continuous functions such that $f(x) \leq g(x)$ holds for all $x \in [a, b]$. Set $A = \{(x, y) \in \mathbb{R}^2 : x \in [a, b]$ and $f(x) \leq y \leq g(x)\}$.

a) Show that $A$ is a closed set (and hence a measurable subset of $\mathbb{R}^2$)

b) If $h : A \to \mathbb{R}$ is a continuous function, then show that $h$ is Lebesgue integrable over $A$ and that 
\[ \int_A h \, d\lambda = \int_a^b \left( \int_{f(x)}^{g(x)} h(x, y) \, dy \right) \, dx \]