1. 
(a) State the Lebesgue Dominated Convergence Theorem.
(b) Use (a) to evaluate
\[ \lim_{n \to \infty} \int_0^1 \frac{dx}{\cos(x + \frac{1}{n}) x^{\frac{3}{2}}} \]
where \( dx \) denotes integration with respect to Lebesgue measure.
[Be sure to explain why the hypotheses are satisfied when you quote (a).]

2. Either prove or provide an explicit counterexample to each of the following assertions: (you may quote without proof familiar relations and containments between \( L^p \)-spaces)
   (a) If \( f, g \in L^2([0, 1]) \) then \( fg \in L^2([0, 1]) \). (Lebesgue measure)
   (b) If \( f, g \in L^2(\mathbb{R}) \) then \( fg \in L^2(\mathbb{R}) \). (Lebesgue measure)
   (c) If \( f, g \in L^2(\mathbb{R}) \) then \( fg \in \ell^2 \). (counting measure)

3. Let \( \lambda \) denote Lebesgue measure on the real line.
   (a) Prove that there is an open set \( \mathcal{O} \) that is dense in \( \mathbb{R} \) with \( \lambda(\mathcal{O}) < 1 \).
   (b) Let \( \mathcal{O} \) be any set satisfying the conclusion to part (a). Prove that \( \mathbb{R} \setminus \mathcal{O} \) is uncountable.
   (c) Let \( \mathcal{O} \) be any set satisfying the conclusion to part (a). Prove that \( \mathbb{R} \setminus \mathcal{O} \) is not compact.

4. Let \( m \) be Lebesgue measure on \([0, 1]\) and \( n \) be counting measure and
   \[ f(x, y) = \begin{cases} 
   1 & \text{if } x = y \\
   0 & \text{if } x \neq y.
   \end{cases} \]
   (a) Show \( \int \int f(x, y)dm(x)dn(y) \neq \int \int f(x, y)dn(y)dm(x) \).
   (b) State the Fubini-Tonelli Theorem and state why the above result does not contradict the Theorem.