METU MATHEMATICS DEPARTMENT
REAL ANALYSIS
SEPTEMBER 2014 - TMS EXAM

1.

(a) State the Lebesgue Dominated Convergence Theorem.

(b) Use (a) to evaluate

$$\lim_{n\to\infty} \int_0^1 \frac{dx}{\cos(x+\frac{1}{n})\, x^{\frac{1}{n}}}$$

where dx denotes integration with respect to Lebesgue measure.

[Be sure to explain why the hypotheses are satisfied when you quote (a).]

2. Either prove or provide an explicit counterexample to each of the following assertions: (you may quote without proof familiar relations and containments between L^p-spaces)

(a) If $f, g \in L^2([0,1])$ then $fg \in L^2([0,1])$. (Lebesgue measure)

(b) If $f, g \in L^2(\mathbb{R})$ then $fg \in L^2(\mathbb{R})$. (Lebesgue measure)

(c) If $f, g \in L^2(\mathbb{R})$ then $fg \in \ell^2$. (counting measure)

3. Let λ denote Lebesgue measure on the real line.

(a) Prove that there is an open set $\mathcal{O}$ that is dense in $\mathbb{R}$ with $\lambda(\mathcal{O}) < 1$.

(b) Let $\mathcal{O}$ be any set satisfying the conclusion to part (a). Prove that $\mathbb{R} \setminus \mathcal{O}$ is uncountable.

(c) Let $\mathcal{O}$ be any set satisfying the conclusion to part (a). Prove that $\mathbb{R} \setminus \mathcal{O}$ is not compact.

4. Let m be Lebesgue measure on $[0,1]$ and n be counting measure and $f(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$

(a) Show $\int\int f(x,y)dm(x)dn(y) \neq \int\int f(x,y)dn(y)dm(x)$.

(b) State the Fubini-Tonelli Theorem and state why the above result does not contradict the Theorem.