16 February 2004

Graduate Preliminary Examination
Real Analysis
Duration: 3 hours

1. Let $(X, \mathcal{M}, \mu)$ be a measure space and $f \in L_1(\mu)$ with $f(x) > 0$ a.e. Prove that if A is a measurable set such that $\int_A f d\mu = 0$, then $\mu(A) = 0$,

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and (a, b) be a finite, non-empty interval in $\mathbb{R}$. Let $f : X \times (a, b) \to \mathbb{R}$ satisfy

a) $F(t) = \int f(x, t)dx$ is defined $\forall t \in (a, b)$

b) $\frac{\partial f}{\partial t}$ is defined everywhere in $X \times (a, b)$

c) There is an integrable $g : X \to [0, \infty)$ such that $|\frac{\partial f}{\partial t}(x, t)| \leq g(x)$ $\forall x \in X$, $t \in (a, b)$.

Prove that both $F'(t)$ and $\int \frac{\partial f}{\partial t}(x, t)dx$ exist $\forall t \in (a, b)$ and are equal.

3. Let $0 < p < q < \infty$ and $(X, \mathcal{M}, \mu)$ be a measure space. Prove that

a) $L^p \not\subset L^q \Leftrightarrow X$ contains sets of arbitrarily small positive measure,

but

b) $\ell_p \underset{\neq}{\subseteq} \ell_q$.

4.

a) State the Lebesgue-Radon-Nikodym Theorem for signed measures.

b) For $j = 1, 2$ let μ_j, ν_j be σ-finite measures on $(X, \mathcal{M}_j)$ s.t. $\nu_j << \mu_j$ (ν_j is absolutely continuous with respect to μ_j). Prove that
$\nu_1 \times \nu_2 << \mu_1 \times \mu_2$ and $\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \cdot \frac{\partial \nu_2}{\partial \mu_2}(x_2)$.