METU - Department of Mathematics
Graduate Preliminary Exam

Real Analysis

February, 2009

Duration: 180 min.

1. Characterize metric spaces X for which the open sets form a σ-algebra.

2. a) Show that if $f : \mathbb{R} \to \mathbb{R}$ is continuous almost everywhere then f is a Lebesgue measurable function.

b) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that the derivative f' of f is a Lebesgue measurable function.

3. Construct an example of a subset $D \subseteq \mathbb{R}^2$ satisfying the following properties:

(i) the Lebesgue measure of $D \cap U$ is zero for every open set $U \subseteq \mathbb{R}^2$;

(ii) the set $D \cap U$ is of the second category for every nonempty open set $U \subseteq \mathbb{R}^2$;

(iii) for every point $(x_0, y_0) \in D$ the line $y = y_0 + x_0 - x$ is a subset of D.

4. a) State the Radon - Nikodym theorem for the interval $[0, 1]$ with the Lebesgue measure.

b) State the Hahn decomposition theorem.

c) Derive the Hahn decomposition theorem for the interval $[0, 1]$ from the Radon - Nikodym theorem and the Jordan decomposition theorem.