# M.E.T.U

## Department of Mathematics

### Preliminary Exam - Feb. 2011

#### REAL ANALYSIS

1. a) Let $\{f_n\}$ be a sequence of measurable functions on a measure space $(X, S, \mu)$ such that $\{f_n(x)\}$ is a bounded sequence for each $x \in X$. Show that the set

$$E = \{x \in X : \lim f_n(x) exists\}$$

is a measurable set.

b) Let $(X, S, \mu)$ be a measure space. Assume that $f : X \to \mathbb{R}$ is a measurable function and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Show that the composition function $gof$ is a measurable function.

2. a) Let $(X, \sum, \mu)$, $(Y, \Lambda, \nu)$ be measure spaces and $\mathcal{A}$ be algebra of subsets of $X \times Y$ generated by rectangles $A \times B$, $A \in \sum$, $B \in \Lambda$. By using of the Monotone Convergence Theorem show that the following function

$$\mu \times \nu : \mathcal{A} \to \ \overline{\mathbb{R}}_+ \ \text{ defined by } \ \mu \times \nu(A \times B) := \mu(A) \cdot \nu(B)$$

is a pre-measure.

b) Let $A = ((a_{ij}))_{ij \in \mathbb{N}}$ be an infinite matrix of real numbers. Suppose $\lim\limits_{i \to \infty} a_{ij} = a_j \in \mathbb{R}$ and $\sup\limits_i |a_{ij}| = b_j$ with $\sum\limits_{j=1}^{\infty} b_j < \infty$. By application of the Dominated Convergence Theorem show that $\lim\limits_{i \to \infty} \sum\limits_{j=1}^{\infty} |a_{ij} - a_j| = 0$

3. a) Formulate Fubini's theorem.

b) Show that if $f(x, y) = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$, with $f(0, 0) = 0$, then

$$\int_0^1 [\int_0^1 f(x, y) dx] dy = -\frac{\pi}{4}, \quad \int_0^1 [\int_0^1 f(x, y) dy] dx = \frac{\pi}{4}$$

c) Can $f$ above be integrable on $[0, 1] \times [0, 1]$ ? Explain.

4. Let $a \in \mathbb{R}$ and $K = \{f \in C^2[0\ 1] : f(0) = f(1) = 0,\ f'(0) = a\}$.

   Find $\min\limits_{f \in k} \displaystyle\int_0^1 (f''(x))^2 dx$ and a function $f \in K$ on which the minimum is attained.

   [Hint: apply Cauchy-Schwartz inequality to functions $\varphi(x) = f''(x), \psi(x) = 1 - x$]